Spectral-Efficiency - Illumination Pareto Front for Energy Harvesting Enabled VLC Systems

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Abstract

The continuous improvement in optical energy harvesting devices motivates the development of visible light communication systems that utilize such available free energy. In this paper, an outdoor VLC system is considered where a VLC base station sends data to multiple users that are capable of harvesting optical energy. The proposed VLC system serves multiple users using time division multiple access (TDMA) with unequal time and power allocation, which are allocated to achieve the system communications and illumination objectives. In an outdoor setup, the system lighting objective is to maximize the average illumination flux, while the communication design objective is to maximize the spectral efficiency (SE). The design objectives are shown to be conflicting, therefore, a multiobjective optimization problem is formulated to obtain the Pareto front of the SE-illumination region. To this end, the marginal optimization problems are solved first using low complexity algorithms. Then, based on the proposed algorithms, a KKT-based algorithm is developed to obtain an inner bound of the Pareto front for the SE-illumination tradeoff. The inner bound for the Pareto-front is shown to be close to the optimal Pareto-frontier via several simulation scenarios for different system parameters.

Index Terms

Visible light communication, spectral efficiency, illumination, energy harvesting, Pareto front, multiobjective optimization, outdoor communication, mass gathering events.

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The SE optimization part of this work was presented in [1].
I. INTRODUCTION

Optical wireless communications (OWC) is a powerful technology that is able to address the spectrum scarcity effectively [2]. It offers different solutions that can meet the exponential increase of wireless communication demand by accessing the infrared, visible and ultraviolet (UV) bands. Free space optical (FSO) communication using infrared laser offers a high speed solution for different applications such as backhaul links. On the other hand, it may suffer from some impairments such as pointing errors, scattering, turbulence and terminal swing. UV or optical scattering communication relaxes the line-of-sight (LoS) requirement in FSO by using wide field-of-view (FOV) receivers and exploiting scattering, while benefiting from the fact that background UV noise from solar radiation is negligible at the ground level. However, the transmitted power should be monitored to satisfy safety requirements.

Among the optical spectrum, the visible light range is of special interest. This owes to the fact that VLC offers high speed data communication thanks to the large modulation bandwidth of the light emitting diodes (LEDs) used recently for illumination. VLC has several advantages over radio frequency (RF) communications, which encourages considering it for next generation wireless networks [3]–[6]. For instance, VLC uses the wide visible light spectrum that extends from a wavelength of 380 nm to 750 nm. Additionally, VLC can benefit from existing infrastructure which makes it cheaper than setting up new RF networks. Furthermore, VLC offers a ubiquitous indoor and outdoor service wherever the illumination service is required. Most research on VLC focused on indoor applications [3], [7]–[11], where a specific illumination level is required.

On the other hand, outdoor VLC has not received much attention except for vehicular applications [12]–[14].

Recently, light energy harvesting-enabled communications systems have gained lots of interest. A multitude of such systems have been investigated in the literature [1], [15]–[19]. In [15], [16] a hybrid VLC/RF relaying communications system with light energy harvesting support was studied where the 1x1 (single transmitter, single receiver) and the 2x2 (two transmitters, two receivers) cases were studied in [15], [16], respectively. In the first setup the authors were interested in the system overall data rate maximization, while in the second setup, the objective was to find the achievable rates region for the two links. In [1], [17], resource allocation problems with the aim of spectral efficiency (SE) maximization for the downlink of a single cell energy harvesting enabled VLC system were considered. By an experimental demonstration, Liu et al.
proposed a solar cell based VLC receiver that is capable of handling both energy harvesting and data decoding [18]. In that work, the developed system was evaluated based on error probability and generated voltage versus illuminance. As the interest in energy harvesting enabled VLC systems increased, Diamantoulakis et al. coined the “SLIPT” term and proposed different strategies for the system operation that hit different points of the trade-off between the amount of harvested energy and communications performance measures for a single user single cell scenario [19].

In this paper, we shed light on one of the promising outdoor VLC applications, namely, serving mass gathering night events e.g., social or sports events. A high degree of illumination is required to support such events. Using VLC in such circumstances should be encouraged for many reasons. First, the absence of sunlight interference, which allows achieving a good performance. Additionally, the high illumination intensity requirement implies the availability of strong signal that is capable of serving a large number of users with high data rates. Moreover, using VLC in such scenarios reduces infrastructure costs as there is no need to set up a new wireless network or use movable base stations, thereby reducing interference and sparing the RF spectrum. Finally, it makes (optical) energy harvesting more promising and viable thanks to the high illumination intensity requirement.

In this paper, we consider an outdoor VLC system with a powerful light source that is used to illuminate a specific area and support delivering data using VLC technology. The illuminated area accommodates a large number of users that can harvest energy from the optical source. We study the problem of allocating the VLC system resources based on a TDMA scheme to maximize the system SE and illumination performance whilst satisfying quality-of-service (QoS) requirements.

The proposed system supports both illumination and communications services, we define QoS in terms of the harvested energy and edge users’ rate. To shed light on an interesting tradeoff that exists in this system, we start by formulating two problems: an SE maximization problem and an illumination maximization problem. Then, we show that the two problems do not follow the same solution structure. This proves that there exists a tradeoff between the two performance metrics. Thus, it becomes important to study the Pareto front of this tradeoff by analyzing the SE-illumination region. Therefore, we consider a multi-objective optimization problem (MOOP) to obtain the best tradeoff curve (Pareto front) between SE and illumination subject to QoS constraints. To this end, we start by solving the marginal problems, i.e., maximizing the SE
and the illumination, by deriving the necessary optimality conditions which are then utilized in proposing low complexity sub-optimal algorithms. Then, we derive the necessary optimality conditions for the MOOP that lead to the optimal SE-illumination Pareto front and propose a low complexity resource allocation algorithm based on the proposed algorithms for the marginal problems. Finally, we present extensive simulations that confirm our proposed algorithms’ capabilities compared with the optimal solution and show the average SE and illumination performance with respect to different system parameters.

The rest of the paper is organized as follows. Section II defines the system model for an outdoor VLC system and different design metrics. Section III demonstrates the tradeoff relation between SE and illumination. Sections IV and V study the marginal optimization problems of the MOOP. Then, the Pareto front solution is investigated in Section VI. Finally, Section VII presents comprehensive numerical results before concluding the paper in Section VIII.

II. SYSTEM MODEL

Consider an outdoor VLC system consisting of a powerful transmitter serving $K$ users using dynamic TDMA with a modulation bandwidth of $B_v$ Hz. All users have visible light energy harvesting capabilities as long as they are in the coverage area. The received signal by the $i$-th user is modeled as

$$y_i = h_i s_i + n_i,$$  \hspace{1cm} (1)

where $s_i$ is the current used to transmit a symbol to the $i$-th user, $h_i$ represents the VLC channel gain between the transmitter and the $i$-th receiver, and $n_i$ is a zero mean additive white Gaussian noise with variance $\sigma^2_n$. The non-negativity of $s_i$ is necessary for the operation of VLC systems, since the driving current of an LED has to be positive.
The transmission frame duration $T$ seconds is divided between users such that user $i$ is allotted $\tau_i T$ seconds during each transmission frame, where $\sum_{i=1}^{K} \tau_i = 1$. Due to practical switching limitations, it is required that $\tau_i \geq \tau_{\text{min}} \ \forall i$. For instance, if a transmission frame of $T$ seconds spans $N$ symbol durations, where the symbol duration is set to be the minimum possible circuits switching time, then user $i$ is alloted 1 symbol duration at least, leading to $\tau_{\text{min}} = T/N$. Due to operational specifications, the average energy consumption at the transmitter per transmitted frame must be equal to $E_v$, i.e., satisfy $T \sum_{i=1}^{K} \tau_i E\{s_i^2\} = E_v$. In this work, we adopt the followed model in [20] where the channel gain $h_i$ between the transmitter and receiver $i$ depends solely on the relative position of the receiver with respect to the transmitter, and is given by:

$$h_i = \eta_{eo} \frac{(m + 1) A_{PD} R_{PD}}{2\pi d_i^2} \frac{\cos^{m+1}(\psi_i)}{\cos^{m+1}(\psi_a)} \text{rect}\left(\frac{\psi_i}{\psi_a}\right),$$

where $m = -\ln 2 / \ln (\cos (\phi_a))$ is the Lambertian order, $\phi_a$ is the semi-angle at half-power of the light source emission pattern, $R_{PD}$ is the photo-detector responsivity $A_{PD}$ is the effective photo-detector area, $d_i$ is the distance between the transmitter and user $i$, $\psi_i$ is the angle between the incident light ray and the normal to the photo-detector plane, $\psi_a$ is the field of view of the user’s receiver, and $\text{rect}(x)$ is the rectangular function defined as $\text{rect}(x) = 1$ if $|x| \leq 1$, and 0 otherwise. We assume perfect knowledge of $h_i \ \forall i$ at the transmitter which is communicated via a feedback mechanism. This assumption is reasonable in a quasi-static channel where $h_i$ maintains the same value throughout a transmission block, and changes in between blocks due to change of location. The electrical to optical conversion efficiency ($\eta_{eo}$) is assumed to be one, and the optical to electrical conversion efficiency is directly proportional to $R_{PD}$, and hence $s_i$ can also be interpreted as optical intensity.

The system QoS requirements are represented as a worst-case user rate requirement and an energy harvesting requirement per user. The former is given by $R_{wc,i} \geq R_{th}$ for some $R_{th}$, where $R_{wc,i}$ is defined as the rate that can be guaranteed for user $i$ when given a fraction $\tau_{\text{min}}$ of time, and located at the same distance from the transmitter as the farthest user. The latter is given by $E_{h,i} \geq \beta P_{ct} \tau_i T$, where $E_{h,i}$ is the amount of harvested energy by user $i$, that must cover a portion $\beta$ of its circuitry energy consumption $P_{ct} \tau_i T$ with $P_{ct}$ being the power used by receiver circuitry (node $i$ is active for a fraction $\tau_i$ of time). The harvested energy is approximated as $E_{h,i} = 0.75 V_T^2 I_o h_i^2 T \sum_{j=1}^{K} \tau_j x_j^2$ for the $i$-th user, where $V_T$ is the thermal voltage (mVolt) and $I_o$ is the dark saturation current of the photo-detector [15]. As for the worst achievable rate, it

\[1\] For mathematical tractability, we use the upper bound on the harvested energy from a VLC cell given in [15].
is defined for the user with the worst link when it uses the shortest time slot duration, i.e., $\tau_{\min}$, as $R_{wc,i} = \frac{R_x \tau_{\min}}{2} \log_2 (1 + \gamma_{\min} x_i^2)$, where $\gamma_{\min} = \min_i \gamma_i$.

In this work we express the overall system SE using the achievable rate for VLC systems presented in [21] with $s_i \sim \text{Exponential}(1/x_i)$ as

$$\eta_{SE}(\mathbf{x}, \mathbf{\tau}) = \frac{1}{2} \sum_{i=1}^{K} \tau_i \log_2 \left(1 + \gamma_i x_i^2\right), \quad (3)$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_K)$ is the average current allocation vector, i.e., $x_i = \mathbb{E}\{s_i\}$, $\mathbf{\tau} = (\tau_1, \tau_2, \ldots, \tau_K)$ is the time fraction allocation vector, and $\gamma_i$ is the $i$-th user channel-to-noise ratio defined as $\gamma_i = \frac{\frac{e^2 h_i^2}{2 \pi}}{\sigma_n^2}$. The system average transmission power is $P_M \triangleq \frac{E_v}{T}$ Watts, i.e., $\sum_{i=1}^{K} \tau_i x_i^2 = P_M$ (since $\mathbb{E}\{s_i^2\} = x_i^2$).

Since the average illumination intensity of the transmitter is proportional to the average LED excitation current intensity, we use the latter quantity as a measure for illumination which can be expressed as follows

$$\mathcal{I}(\mathbf{x}, \mathbf{\tau}) = \sum_{i=1}^{K} \tau_i x_i. \quad (4)$$

The previously mentioned metrics (SE and illumination intensity) evaluate the outdoor VLC system performance in terms of communication and lighting. It is desirable for outdoor VLC setups to maximize these metrics which will be shown to be conflicting in the next section.

III. MOTIVATION TO MULTI-OBJECTIVE OPTIMIZATION: A TOY EXAMPLE

In this section, we shed the light on the existence of a trade-off between illumination functionality and spectral efficiency of the downlink VLC system. For this purpose, we consider a two-user toy example with the time and power constraints that are written as

$$\tau_1 + \tau_2 = 1, \quad (5)$$

$$\tau_1 x_1^2 + \tau_2 x_2^2 = P_M. \quad (6)$$

Firstly, consider the the illumination maximization problem, which can be expressed by substituting (5) and (6) in (4) as

$$\max_{x_1, \tau_1} \tau_1 x_1 + \sqrt{1 - \tau_1} \sqrt{P_M - \tau_1 x_1^2}. \quad (7)$$

The stationarity conditions, that are necessary for optimality, are found by taking the derivative of the objective function in (7) with respect to $\tau_1$ and $x_1$, respectively, then equating the result to zero leading to
\[
x_1 - \frac{\sqrt{P_M - \tau_1 x_1^2}}{2\sqrt{1 - \tau_1}} + \sqrt{1 - \tau_1} - \frac{x_1^2}{2\sqrt{P_M - \tau_1 x_1^2}} = 0 \quad (8)
\]
\[
\tau_1 + \frac{-2\tau_1 x_1 \sqrt{1 - \tau_1}}{2\sqrt{P_M - \tau_1 x_1^2}} = 0. \quad (9)
\]
One can show from (9) that \( x_1 = \sqrt{P_M} \) for any \( \tau_1 \in [0, 1] \) is a necessary condition for the illumination maximization.

Secondly, the SE maximization and can be expressed by substituting (5) and (6) in (3) giving
\[
\max_{x_1, \tau_1} 0.5\tau_1 \log_2 \left( 1 + \gamma_1 x_1^2 \right) + 0.5 \left( 1 - \tau_1 \right) \log_2 \left( 1 + \frac{P_M - \tau_1 x_1^2}{1 - \tau_1} \right).
\]
The stationarity conditions of the SE maximization problem are found in a similar way from
\[
\frac{\gamma_2 (P_M - x_1^2)}{\gamma_2 (P_M - \tau_1 x_1^2) + 1 - \tau_1} - \ln \left( \frac{\gamma_2 (P_M - \tau_1 x_1^2)}{1 - \tau_1} + 1 \right) + \ln \left( 1 + \gamma_1 x_1^2 \right) = 0 \quad (11)
\]
\[
2\tau_1 x_1 \left( \frac{\gamma_1}{1 + \gamma_1 x_1^2} - \frac{\gamma_2 (1 - \tau_1)}{\gamma_2 (P_M - \tau_1 x_1^2) + 1 - \tau_1} \right) = 0. \quad (12)
\]
Now, to find out whether there is a solution that maximizes both objectives, i.e. illumination and SE, we find the intersection between stationary points space of both problems. By substituting the obtained potential illumination maximization points \( x_1 = \sqrt{P_M} \) in (11) we get:
\[
- \ln \left( 1 + \gamma_2 P_M \right) + \ln \left( 1 + \gamma_1 P_M \right) = 0,
\]
which can not hold since \( \gamma_1 \neq \gamma_2 \) (almost surely), therefore, the optimal illumination solution does not maximize the system overall SE, which confirms the trade-off relation between the two design objectives. Hence, MOOP is the natural optimization approach to deal with these conflicting objective functions. The MOOP of SE and illumination is expressed for a given priority factor \( \alpha \) with \( \alpha \in [0, 1] \) as,
\[
\max_{x, \tau} \alpha \eta_{SE}(x, \tau) + (1 - \alpha) I (x, \tau) \quad (14)
\]
To solve (14) in a tractable way, we study first the marginal optimization problems, then we solve the MOOP problem using the proposed solutions for the marginal problems. This procedure is detailed in the following sections.

IV. SPECTRAL EFFICIENCY OPTIMIZATION

In this section, we study the resource allocation problem of the proposed VLC system to maximize the SE. As a result, we formulate the optimization problem as follows

(P1) \[
\max_{x, \tau} \eta_{SE}(x, \tau)
\]
subject to \[
C1: \sum_{i=1}^{K} \tau_i x_i^2 = P_M, \quad C2: \sum_{i=1}^{K} \tau_i = 1, \quad C3: E_{h,i} \geq \beta P_{cr} \tau_i \quad \forall i
\]
\[
C4: \tau_i \geq \tau_{\min} \quad \forall i, \quad C5: R_{wc,i} \geq R_{th} \forall i,
\]
where \( C_1 \) reflects the power budget constraint, \( C_2 \) denotes the time allocation constraint, \( C_3 \) is the energy harvesting constraint, \( C_4 \) states the minimum time slot according to hardware switching limitations and \( C_5 \) is the QoS constraint in terms of the worst link rate.

To solve \((P_1)\), we first simplify the constraints. \( C_3 \) can be rewritten equivalently based on \( C_1 \) as \( \tau_i \leq \frac{0.75 \sqrt{V T o h}}{P_M} \triangleq \tau_{\text{max},i} \). Moreover, \( C_5 \) can be written equivalently as \( x_i \geq \sqrt{\frac{2 R_{\text{th}}}{(B v \tau_{\text{min}})} - 1} / \gamma_{\text{min}} \triangleq x_{\text{min}} \). Then, we note that \((P_1)\) is not a convex optimization problem since \( \eta_{\text{SE}}(x, \tau) \) is not concave in \( x \) and \( \tau \). Therefore, to convexify the problem, we reformulate \((P_1)\) using the transformation \( z_i = \tau_i x_i^2 \) to obtain

\[
(P_1) \quad \max_{z, \tau} \quad \bar{\eta}_{\text{SE}}(z, \tau) = \sum_{i=1}^{K} \tau_i \ln \left( 1 + \frac{\gamma_i z_i}{\tau_i} \right)
\]

subject to

\[
C_1 : \sum_{i=1}^{K} z_i = P_M, \quad C_2, \quad C_4
\]

\[
C_3 : \tau_i \leq \tau_{\text{max},i} \forall i, \quad C_5 : z_i \geq z_{\text{min}} \forall i,
\]

where \( z = (z_1, z_2, \ldots, z_K) \) is the power allocation vector, and \( z_{\text{min}} = \tau_{\text{min}} x_{\text{min}}^2 \). One can show that \( \bar{\eta}_{\text{SE}}(z, \tau) \) is a jointly concave function in both \( z \) and \( \tau \) using the perspective transform of \( \ln (1 + \gamma_i z_i) \) which is a concave function in \( z \) [22]. In addition, all the constraints are affine, thus, \((P_1)\) is a convex optimization problem and the Karush-Kuhn-Tucker (KKT) conditions gives global optimality [22].

The Lagrangian formulation of \((P_1)\) is given by:

\[
L = \sum_{i=1}^{K} \tau_i \ln \left( 1 + \frac{\gamma_i z_i}{\tau_i} \right) + \mu \left( P_M - \sum_{i=1}^{K} z_i \right) + \sum_{i=1}^{K} o_i (z_i - z_{\text{min}}) + \sum_{i=1}^{K} \nu_i (\tau_{\text{max},i} - \tau_i)
\]

\[
+ \lambda \left( 1 - \sum_{i=1}^{K} \tau_i \right) + \sum_{i=1}^{K} \kappa_i (\tau_i - \tau_{\text{min}}).
\]

The KKT conditions of \((\tilde{P}_1)\) can be summarized as

\[
\frac{\partial L}{\partial z_i} = \frac{\gamma_i}{1 + \gamma_i z_i/\tau_i} - \mu + o_i = 0, \quad (16)
\]

\[
\frac{\partial L}{\partial \tau_i} = \ln \left( 1 + \frac{\gamma_i z_i}{\tau_i} \right) - \frac{\gamma_i z_i}{\tau_i} + \gamma_i z_i - \nu_i - \lambda + \kappa_i = 0, \quad (17)
\]

\[
o_i (z_i - z_{\text{min}}) = \nu_i (\tau_i - \tau_{\text{max},i}) = \kappa_i (\tau_i - \tau_{\text{min}}) = 0, \quad (18)
\]

\[
o_i \geq 0, \quad \nu_i \geq 0, \quad \kappa_i \geq 0 \quad (19)
\]

\( \forall i \) in addition to the conditions \( C_1 \sim C_5 \). The convexity of \((\tilde{P}_1)\) implies that strict complementary slackness applies [22], such that \( o_i = 0 \) if and only if \( z_i \geq z_{\text{min}} \) and \( o_i > 0 \) iff \( z_i = z_{\text{min}} \).
The objective function in $\nu$ to develop low complexity algorithms with performance close to the optimal solution.

$K$ following structure:

where $z$ and $\nu$ constraints associated with $\gamma_i$ and $\kappa_i$ (corresponding to zero or nonzero values of $o_i$, $\kappa_i$, and $\nu_i$), which results in numerous possible KKT stationarity cases that are summarized in Table I, where $\tau_i^*$ and $z_i^*$ denote the optimal values of $\tau_i$ and $z_i$, respectively. Hence, solving the resource allocation problem turns out to be of exponential complexity in terms of $K$ since we will need to try different combinations of the 6 Cases listed in Table I for each user. Therefore, it is desirable to develop low complexity algorithms with performance close to the optimal solution.

In order to reduce the complexity, we make use of the positive monotonicity characteristics of the objective function in $\tau$ and $z$ (cf. Appendix A) and limit our space of solutions to the following structure:

$$z = [z_1^*, \ldots, z_j^*, z_{\min}, \ldots, z_{\min}]$$

$$\tau = [\tau_{\max,1}, \ldots, \tau_{\max,f-1}, \tau_{\ell}^*, \ldots, \tau_{\ell}^*, \tau_{\min}, \ldots, \tau_{\min}]$$

where $z_i^* > z_{\min}$ for $i \leq j$ and $\tau_{\max,f-1} > \tau_{\ell}^* > \tau_{\min}$ for $f \leq i \leq \ell$ with $N_\tau$ unknown time allocation variables, where $N_\tau = \ell - f + 1$. Now, we consider the proposed solution structure and different cases in Table I to allocate the available resources. First, one can easily show that there are no more than two users can fall into Case 1. Then, we need to check different possibilities that meet the aforementioned criteria to find the feasible solution with the best performance. To scan all possibilities, we consider two scenarios $N_\tau = 1$ and $N_\tau > 1$ as follows:

**Scenario A:** If $N_\tau = 1$, then $f = \ell$ and $\tau_{\ell}^*$ is obtained from C2 as $\tau_{\ell}^* = 1 - \sum_{i=1}^{f-1} \tau_{\max,i} - \ldots$
Algorithm I-A

1: for \( \ell \in \{KS + (K - 1)(1 - S), \ldots, 2, 1\} \) 
2: \( t \leftarrow \sum_{i=1}^{\ell-1} \tau_{\max,i} + (K - \ell) \tau_{\min S} + (1 - S) \sum_{i=\ell+1}^{K} \tau_{\max,i} \) 
3: if \( t < 1 \) and \( \tau_{\min} < 1 - t < \tau_{\max,\ell} \) 
4: \( \tau_{\ell} \leftarrow 1 - t \) and \( \tau_i \leftarrow \tau_{\max,i}, \forall i \leq \ell - 1 \) 
5: \( \tau_i \leftarrow \tau_{\min S} + (1 - S) \tau_{\max,i}, \forall i \geq \ell + 1 \) 
6: \( j \leftarrow K \) 
7: while \( j \geq 1 \) 
8: \( z_i \leftarrow z_{\min}, \forall i > j \) 
9: if \( A=0 \) 
10: \( z_i \leftarrow \frac{r_{\min}^{\gamma_i^*}}{4\mu^2}, \forall i \leq j \) with \( \mu \) obtained from (64). 
11: else 
12: \( z_i \leftarrow \tau_i \left( \frac{1}{\mu} - 1/\gamma_i \right) \) \( \forall i \leq j \) with \( \mu \) obtained from (22). 
13: end if 
14: Execute Algorithm I-KKT 
15: \( j \leftarrow j - 1 \) 
16: end while 
17: end if 
18: end for

\[ \tau_{\min} (K - \ell). \] To obtain \( z_i^* \) for \( i \leq j \), we compute it using the corresponding time allocation as 
\[ z_i^* = \tau_i \left( \frac{1}{\mu} - \frac{1}{\gamma_i} \right), \] which captures Cases 1, 2 and 3 in Table I. As for \( \mu \), it is obtained using C1 as 
\[ \mu = \frac{\sum_{i=1}^{j} \tau_i}{\sum_{i=1}^{j} \frac{\tau_i}{\gamma_i} + P_M - (K - j) z_{\min}}. \] (22)

It remains to specify \( j \) and \( \ell \). To do this, we search for \( j \in \{1, \ldots, K\} \) and \( \ell \in \{1, \ldots, K\} \) using Algorithm I-A\(^2\) and check if the KKT conditions are satisfied using Algorithm I-KKT.

**Scenario B:** If \( N_r > 1 \), then the proposed solution structure and Case 1 condition impose \( j \leq f + 1 \). To handle this scenario, we divide the \( j \) domain into three sub-scenarios given by

\(^2\)The proposed algorithms are developed to deal with both SE maximization and MOOP by using the parameter \( A \) that takes either 1 or 0 values, respectively, to select the optimization problem.
Algorithm I-KKT

1: if $z$ and $\tau$ are feasible, and $\tilde{\eta}_{SE}(z, \tau) \geq \eta_{SE}^{\text{max}}$
2: $z^* \leftarrow z$, $\tau^* \leftarrow \tau$, $\eta_{SE}^{\text{max}} \leftarrow \tilde{\eta}_{SE}(z, \tau)$
3: end if
4: if $A=0$
5: Compute $o_i$, $\kappa_i$, and $\nu_i$ using (54) and (55) $\forall i$
6: else
7: Compute $o_i$, $\kappa_i$, and $\nu_i$ using (16) and (17) $\forall i$
8: end if
9: if KKT conditions are satisfied
10: $z^* \leftarrow z$, $\tau^* \leftarrow \tau$, Terminate.
11: end if

$j = f$, $j = f + 1$ and $j < f$. Before considering the different sub-scenarios, we define $U_x$ as the set of user indices belonging to Case $x$ in Table I.

Scenario B.i: for $j = f$, $z^*_i$ follows Cases 1 and 2 in Table I for $i \leq j$ and is expressed as

$$z^*_i = \begin{cases} \tau_{\text{max},i} \left( \frac{1}{\mu} - \frac{1}{\gamma_i} \right) \triangleq f_i(\mu), & \forall i \in U_2 \\ \tau_i^* \left( \frac{1}{\mu} - \frac{1}{\gamma_i} \right), & \forall i \in U_1, \end{cases}$$

(23)

where $U_2 \triangleq \{1, \ldots, j-1\}$ and $U_1 \triangleq \{j\}$. As for $i \geq j + 1$, $z_i = z_{\text{min}}$ according to the proposed structure in (20). On the other hand, $\tau_i^*$ is expressed as

$$\tau_i^* = z_{\text{min}} \gamma_i \left( \frac{1}{r_i} - 1 \right)^{-1}, \forall i \in U_4,$$

(24)

where $U_4 \triangleq \{j + 1, \ldots, \ell\}$ and $r_i = \mu - \frac{\alpha_i}{\tau_i}$, while $\tau_i = \tau_{\text{min}}$ for $\ell + 1 \leq i \leq K$.

To evaluate $z_i^*$ from (23) and $\tau_i^*$ from (24), we need to find $\mu$ and $\alpha_i \forall i \in U_4$. Thus, we first rewrite the second equation presented under Case 4 in Table I in terms of $r_i$ obtaining

$$r_i = -W_0 \left( -e^{-(\lambda+1)} \right),$$

(25)

where $W_0(.)$ is the Lambert function [23]. This relation can proved as follows:

To solve $-\ln (r_i) - (1 - r_i) - \lambda = 0$, we rewrite it equivalently as $-\ln (r_i e^{-r_i}) = \lambda + 1$, which can simplified to

$$- r_i e^{-r_i} = -e^{-(\lambda+1)}.$$

(26)
Then, by computing LambertW function to both sides of (26) we get $-r_i = \mathcal{W} \left( -e^{-(\lambda+1)} \right)$. Since $-r_i \geq -1$ then $r_i = -\mathcal{W}_0 \left( -e^{-(\lambda+1)} \right)$. Thus, $\tau^*_i$ in (24) can be rewritten in terms of $\lambda$ as

$$
\tau^*_i = z_{\min} \gamma_i \left( \frac{1}{-\mathcal{W}_0 \left( -e^{-(\lambda+1)} \right)} - 1 \right)^{-1} \triangleq g_i(\lambda), \quad \forall i \in U_4. \quad (27)
$$

In addition, $\tau^*_j$ can be expressed in terms of $\lambda$ using $C2$ as,

$$
\tau^*_j = 1 - \sum_{i \in U_4} g_i(\lambda) - \sum_{i=1}^{f-1} \tau_{\max,i} - (K-\ell)\tau_{\min}. \quad (28)
$$

Since user $j$ follows Case 1, then we can obtain $\mu$ by solving Case 1 second equation at $i = j$ obtaining

$$
\mu = -\gamma_j \mathcal{W}_0 \left( -e^{-(\lambda+1)} \right). \quad (29)
$$

Then, we use $C1$ and substitute (27)–(29) in (23) obtaining the following equation in $\lambda$

$$
G(\lambda) = \left( 1 - \sum_{i=1}^{f-1} \tau_{\max,i} - (K-\ell)\tau_{\min} - \sum_{i \in U_4} g_i(\lambda) \right) \left( \frac{1}{-\gamma_j \mathcal{W}_0 \left( -e^{-(\lambda+1)} \right)} - \frac{1}{\gamma_j} \right) + \sum_{i \in U_2} f_i \left( -\gamma_j \mathcal{W}_0 \left( -e^{-(\lambda+1)} \right) \right) = P_M - (K-j)z_{\min}. \quad (30)
$$

The solution of (30) can be obtained by using the bisection numerical method thanks to the monotonic behavior of $G(\lambda)$, which admits a unique solution (cf. Appendix B). By backward substitution, we can get all the unknown primal variables, and then the dual variables can be calculated using (16) and (17). This is illustrated in lines 13–15,19,29 in Algorithm I-B.

**Scenario B.ii:** For $j = f + 1$, we have $U_2 \triangleq \{1, \ldots, j-2\}$, $U_1 \triangleq \{j-1, j\}$ and $U_4 \triangleq \{j + 1, \ldots, \ell\}$. $z^*_i, \forall i \in U_1$ or $U_2$ is evaluated from (23) and $\tau^*_i, \forall i \in U_4$ is evaluated from (27). Similar to our approach in Scenario B.i, we need to know $\mu$ and $\lambda$ to compute $z^*_i$ and $\tau^*_i$. For this purpose, we consider the two users in $U_1$ and use the second equation of Case 1 in Table I for both users to find $\mu$ as

$$
\mu = \ln \left( \frac{\gamma_j}{\gamma_{j-1}} \right) - \frac{1}{\gamma_j - 1}. \quad (31)
$$

Moreover, we evaluate the same equation for user $j$ to find $\lambda$ as

$$
\lambda = \ln \left( \frac{\gamma_j}{\mu} \right) - 1 + \frac{\mu}{\gamma_j}. \quad (32)
$$

Now, we substitute (23) and (32) in C1 to obtain $\tau^*_j$ and $\tau^*_{j-1}$ that can be computed as

$$
\begin{bmatrix}
\tau^*_j \\
\tau^*_{j-1}
\end{bmatrix} = \begin{bmatrix}
(1/\mu - 1/\gamma_j) & (1/\mu - 1/\gamma_{j-1}) \\
1 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
P_M - z_{\min}(K-j) - \sum_{i \in U_2} z^*_i \\
1 - \tau_{\min}(K-\ell) - \sum_{i \in U_4} \tau_{\max,i}
\end{bmatrix}. \quad (33)
$$
Finally, the dual variables can be computed from (16), (17) as shown in Algorithm I-KKT in lines 5 and 7.

**Scenario B.iii:** for \( j < f \), we have \( U_1 \triangleq \emptyset \), \( U_2 \triangleq \{1, \ldots, j\} \) and \( U_4 \triangleq \{j+1, \ldots, f\} \). \( z_i^*, \forall i \in U_2 \) is evaluated from (23) where \( \mu \) is computed from C1 as

\[
\mu = \frac{\sum_{i \in U_2} \tau_{\text{max},i}}{\sum_{i \in U_2} \gamma_i} + P_M - (K - j) z_{\text{min}}.
\]  

(34)

As for \( \tau_i^*, \forall i \in U_4 \), it is evaluated from (27), where \( \lambda \) is found from C2. Afterwards, all unknown \( \tau \) variables are obtained. We then calculate the rest of dual variables using (16) and (17). This is illustrated in lines 5 and 7 in Algorithm I-KKT.

In summary, Algorithm I proceeds through three composite stages. The first stage scans values of \( N_\tau \) that is initially set to 1 and is incremented by one after each failed iteration (when KKT conditions are not satisfied). In the second stage, we look over all possible \( f \) values which yields a feasible time allocation (lines 1–5 in Algorithms I-A and I-B). For every feasible \( f \) value, the third stage loops over the possible configurations for \( z \) variables (\( j \) values). For each computed solution, the primal and dual variables are calculated, the solution feasibility is checked and the solution is compared to the best feasible solution found so far \( \eta_{\text{SE}}^{\text{max}} \) (lines 6 onwards in Algorithms I-A and I-B). If the current solution has a better objective function value than \( \eta_{\text{SE}}^{\text{max}} \), we save the current solution and update \( \eta_{\text{SE}}^{\text{max}} \).

A. Reduced Complexity Algorithms

Throughout this subsection, we present two simpler algorithms than Algorithm I to find a solution that maximizes the SE while meeting the required QoS. As for the complexity of Algorithm I, it depends on number of users, i.e., \( K \). Algorithm I consists of three composite iterative stages where the worst case complexity of each is \( \mathcal{O}(K) \), resulting in an overall complexity of \( \mathcal{O}(K^3) \). One way to reduce the complexity is to force \( N_\tau \) to be 1, which guarantees finding a feasible solution. As a result, the proposed reduced complexity algorithm or **Algorithm II** has two composite iterative stages that leads to a complexity of \( \mathcal{O}(K^2) \). Algorithm II is realized by modifying line 4 of Algorithm I to be \( N_\tau \leq 1 \) instead of \( N_\tau \leq K \).

To reduce complexity further to linear complexity in \( K \), we adopt **Algorithm III** to allocate the available resources. In this algorithm, the time budget is poured such that all users at first are allocated the minimum amount of time \( \tau_{\text{min}} \), then the remaining budget \( (1-K\tau_{\text{min}}) \) is distributed iteratively such that users are allocated their time portions in descending order of channel gains,
Algorithm I-B

1: for $\ell = \{KS + (K - 1)(1 - S)(1 - A), \ldots, N_\tau + 1, N_\tau\}$
2: $t \leftarrow \sum_{i=1}^{\ell-N_\tau} \tau_{\text{max},i} + (K - \ell) \tau_{\text{min}} S + (1 - S) \sum_{i=\ell+1}^{K} \tau_{\text{max},i}$
3: $U_4 \leftarrow \{\ell + 1, \ldots, K\}$
4: if $t < 1$ and $(K - \ell) \tau_{\text{min}} S + (1 - S) \sum_{i=\ell+1}^{K} \tau_{\text{max},i} < 1 - t < \sum_{i=\ell-N_\tau+1}^{\ell} \tau_{\text{max},i}$
5: $\tau_i \leftarrow \tau_{\text{max},i}, \forall i \leq \ell - 1$ and $\tau_i \leftarrow \tau_{\text{min}} S + \tau_{\text{max},i} (1 - S), \forall i \geq \ell + 1$
6: $j \leftarrow \ell - N_\tau + 1 + A$
7: while $j \geq 1$
8: $z_i \leftarrow z_{\text{min}}, \forall i > j$
9: if $A = 1$
10: if $j = \ell - N_\tau + 2$
11: $U_2 \leftarrow \{1, \ldots, j - 2\}$
12: Compute $\mu$ from (31), Then get $\lambda$ from (32), $\tau_j, \tau_{j-1}$ from (33), $z_j, z_{j+1}$ from (23)
13: else if $j = \ell - N_\tau + 1$
14: $U_2 \leftarrow \{1, \ldots, j - 1\}$, Solve (29) for $\mu$ and (30) for $\lambda$
15: Compute $z_j$ and $\tau_j$ from (23) and (28), respectively
16: else
17: $U_2 \leftarrow \{1, \ldots, j\}$, Solve (28) for $\lambda$ with $\tau_j = 0$, Compute $\mu$ from (34)
18: end if
19: Compute $z_i$ from (23) $\forall i \in U_2$, $\tau_i$ from (27) $\forall i \in U_4$.
20: else $\%A = 0$
21: if $j = \ell - N_\tau + 1$
22: $U_2 \leftarrow \{1, \ldots, j - 1\}$, Compute $\lambda$ from (56), then Compute $\mu$ from (57)
23: Compute $z_j$ and $\tau_j$ from (60) and (61), respectively.
24: else
25: Compute $\lambda$ from (62), $\mu$ from (63)
26: end if
27: Compute $\tau_i$ from (59) $\forall i \in U_4$, $z_i$ from (58) $\forall i \in U_2$
28: end if
29: Execute Algorithm I-KKT , $j \leftarrow j - 1$
30: end while
31: end if
32: end for
Algorithm I: SE optimization / SE-illumination MOOP

1: **Input** \( A, P_M, z_{\min}, \tau_{\min}, \beta, \gamma_i, \tau_{\max,i} \forall i \in \{1, \ldots, K\} \) % \( A=1 \) or 0 for max SE or MOOP, respectively

2: **Initialize** \( \eta_{SE}^{\max} = 0, N_\tau = 1 \)

3: **for** \( S = 1, \ldots, A \)

4: **while** \( N_\tau \leq K \)

5: **if** \( N_\tau = 1 \)

6: **Execute** Algorithm I-A

7: **else**

8: **Execute** Algorithm I-B

9: **end if**

10: \( N_\tau \leftarrow N_\tau + 1 \)

11: **end while**

12: \( N_\tau \leftarrow 1, l \leftarrow K - 1, j \leftarrow K \)

13: **end for**

where in the \( i \)-th iteration, user \( i \) is allocated the minimum between the remaining time budget and the maximum time portion \( \tau_{\max,i} \). As for the power, it is assumed to be allocated uniformly among users. Therefore, the complexity performance of the proposed algorithm become \( O(K) \).

Algorithm III: SE optimization

1: **Input** \( \tau_{\min}, \tau_{\max,i} \forall i \in \{1, \ldots, K\} \)

2: **Initialize** \( j = l = K, n = 1 \), where \( n \equiv N_\tau \)

3: **for** \( l = K : -1 : 1 \)

4: \( t \leftarrow \sum_{i=1}^{\ell-1} \tau_{\max,i} + (K - l) \tau_{\min} \)

5: **if** \( t < 1 \) and \( \tau_{\min} < 1 - t < \tau_{\max,l} \)

6: \( \tau_i \leftarrow \tau_{\max,i}, \forall i \leq l - 1 \) and \( \tau_i \leftarrow \tau_{\min}, \forall i \geq l + 1 \)

7: \( \tau_\ell \leftarrow 1 - t, z_i \leftarrow P_M/K \)

8: **end if**

9: **end for**
B. Problem Feasibility

By considering the constraints $C_1 - C_5$ it can be shown that the following conditions are necessary for the considered optimization problems feasibility:

\[
\min_i \tau_{\text{max},i} \geq \tau_{\text{min}} \quad (35)
\]

\[
\sum_{i=1}^K \tau_{\text{max},i} \geq 1 \quad (37)
\]

\[
K \tau_{\text{min}} \leq 1 \quad (36)
\]

\[
K z_{\text{min}} \leq P_M. \quad (38)
\]

Condition (35) is necessary to maintain the problems feasible from $C_3$ and $C_4$ perspective. Moreover, condition (36) is necessary for problem feasibility considering $C_2$ and $C_4$ constraints. In addition, it can be deduced that condition (37) is necessary for $C_2$ and $C_3$ constraints and (38) is needed to satisfy $C_1$ and $C_5$.

The aforementioned feasibility condition set can give bounds for $K$. Specifically, (36) and (38) can provide an upper bound on the number of users ($K \leq \min(P_M/z_{\text{min}}, 1/\tau_{\text{min}})$) that can be accommodated given the existing channel conditions and the required data transmission rate. On the other hand, a lower bound on $K$ can be deduced from (37).

V. ILLUMINATION MAXIMIZATION

In this section, we study the illumination maximization problem subject to $C_1 - C_5$. To this end, we formulate the illumination maximization problem by apply the same transformation of variables, $z_i = \tau_i x_i^2 \forall i$, on the illumination objective function (4) and the constraints giving

\[
\text{(P2)} \quad \max_{z_i, \tau} \quad \sum_{i=1}^K \sqrt{z_i \tau_i}
\]

subject to $C_1 - C_5$.

It can be easily proven that \text{(P2)} is a convex optimization problem, as the objective function is the positive weighted sum of concave functions [22]. The function $\sqrt{z_i \tau_i}$ represents the perspective transform of $\sqrt{z_i}$ with respect to $\tau_i$, which confirms the concavity of $\sqrt{z_i \tau_i}$ [22].

The Lagrangian of problem \text{(P2)} can be expressed as

\[
\mathcal{L}_1 = \sum_{i=1}^K \sqrt{\tau_i z_i} + \mu \left( P_M - \sum_{i=1}^K z_i \right) + \sum_{i} o_i \left( z_i - z_{\text{min}} \right) + \lambda \left( 1 - \sum_{i=1}^K \tau_i \right) \\
+ \sum_{i=1}^K \nu_i \left( \tau_{i,\text{max}} - \tau_i \right) + \sum_{i=1}^K \kappa_i \left( \tau_i - \tau_{\text{min}} \right). \]

(39)
TABLE II: Inequality constraints multipliers configurations for \((\tilde{P}_2)\). The optimal \((\tau_i, z_i)\) pair is indicated below each case.

<table>
<thead>
<tr>
<th>(\nu_i = 0)</th>
<th>(o_i = 0)</th>
<th>(\nu_i \neq 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1 ((\tau^<em>_i, z_i^</em>))</td>
<td>(\frac{\tau^<em>_i}{z_i^</em>} = \frac{1}{4\mu} ) and (\frac{1}{4\mu} - \lambda = 0)</td>
<td>Case 3 ((\tau^<em><em>i, z</em>{\min}^</em>))</td>
</tr>
<tr>
<td>(\frac{z_i}{\tau_{\max,i}} = \frac{1}{4\mu} ) and (\frac{1}{4\mu} - \nu_i - \lambda = 0)</td>
<td>(\nu_i \neq 0)</td>
<td></td>
</tr>
<tr>
<td>Case 2 ((\tau_{\max,i}, z_i^*))</td>
<td>Case 4 ((\tau_{\max,i}, z_{\min}^*))</td>
<td></td>
</tr>
<tr>
<td>(\frac{z_{\min}}{\tau_{\max,i}} = \frac{1}{4\mu} ) and (\frac{1}{4\mu} - \nu_i - \lambda = 0)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The KKT conditions of this problem can be summarized as:

\[
\frac{\partial L_I}{\partial z_i} = \frac{1}{2} \sqrt{\tau_i} - \mu + o_i = 0 \quad \forall i \in \{1, \ldots, K\},
\]

\[
\frac{\partial L_I}{\partial \tau_i} = \frac{1}{4} \frac{1}{(\mu - o_i)} - \nu_i - \lambda + \kappa_i = 0 \quad \forall i \in \{1, \ldots, K\},
\]

in addition to C1 – C5, (19) and (18).

The convexity of problem \((\tilde{P}_2)\) implies strict complementary slackness, where \(o_i = 0\) iff \(z_i > z_{\min}\) and \(o_i > 0\) iff \(z_i = z_{\min}\) \(\forall i\). Similarly, \(\kappa_i = 0\) iff \(\tau_i > \tau_{\min}\), \(\kappa_i > 0\) iff \(\tau_i = \tau_{\min}\), \(\nu_i = 0\) iff \(\tau_i < \tau_{\max,i}\), and \(\nu_i > 0\) iff \(\tau_i = \tau_{\max,i}\), \(\forall i\). To avoid the exponential complexity of solving the KKT system due to presence of inequality constraints, we limit our space of solutions to specific structure(s). In the following, we propose a structure inspired by the impact of different constraints.

To find a good structure for the solution, we first consider \((\tilde{P}_2)\) with C1, and C2 only. The solution of the simplified problem reduces to \(z_i^* = P_M \tau_i^*\), which can be proven using Cauchy-Shwartz inequality\(^3\). After adding C3 – C5, we have \(\tau_{\min} \leq \tau_i \leq \tau_{\max,i}\), which may violate the proposed structure if \(z_{\min} > P_M \min_1 \tau_{\max,i}\). One way to find a solution with a close structure to \(z_i^* = P_M \tau_i^*\) while satisfying the constraints is to allot the marginal resources to some users, i.e., \(\tau_{\max,i}\) and \(z_{\min}\), while others should be computed according to the KKT conditions. Consequently, we limit our search space to the following format:

\[
z^* = [z_1^*, \ldots, z_\ell^*, z_{\min}, \ldots, z_{\min}] \quad \text{and} \quad \tau^* = [\tau_1^*, \ldots, \tau_j^*, \tau_{\max,j+1}, \ldots, \tau_{\max,K}] \quad (42)
\]

\(^3\)It is worthy to note that uniform allocation solution \((z_i = P_M/K, \tau_i = 1/K \forall i)\) is one of the solutions that follow this structure.
Based on the adopted solution structure and the KKT conditions, all possibilities of the associated inequality multipliers are listed in Table II. As for the indices, there are two possible scenarios either \( j \leq \ell \) or \( j > \ell \) as discussed in the sequel.

**Scenario A:** when \( j \leq \ell \), \( z_i^* \) is expressed in terms of \( \tau_i^* \) for \( i \leq j \) according to Case 1 as

\[
 z_i^* = 4\lambda^2 \tau_i^* \quad \forall i \in U_1, \tag{44}
\]

where \( U_1 \triangleq \{1, \ldots, j\} \). For \( j < i \leq \ell \), \( z_i^* \) is found based on Case 2 as

\[
 z_i^* = 4\lambda^2 \tau_{\max,i} \quad \forall i \in U_2, \tag{45}
\]

where \( U_2 \triangleq \{j+1, \ldots, \ell\} \). Now, \( \lambda \) can be found from C1 and C2 as

\[
 \lambda = \frac{1}{2} \sqrt{\frac{P_M - (K - \ell) z_{\min}}{1 - \sum_{i \in U_4} \tau_{\max,i}}}, \tag{46}
\]

where \( U_4 \triangleq \{\ell+1, \ldots, K\} \).

It can be proved that for a given configuration of \( \ell \) and \( j \) to be feasible, the following conditions should be met

\[
 \tau_{\min} \leq \frac{z_{\min}}{4\lambda^2} \leq \min_{i \in U_2} \tau_{\max,i} \quad \text{and} \quad j \frac{z_{\min}}{4\lambda^2} \leq 1 - \sum_{i \in U_4} \tau_{\max,i} - (\ell - j) \frac{z_{\min}}{4\lambda^2} \leq \sum_{i \in U_1} \tau_{\max,i}. \tag{47}
\]

For a given feasible choice of \( \ell, j \), \( \tau_i^* \) where \( i \in U_1 \) are selected to satisfy the feasibility conditions, then the rest of \( \tau_i^* \) variables and \( z_i^* \) are computed as discussed previously, and the optimality is examined by finding \( \nu_i, o_i \) \( \forall i \) from (40) and (41).

**Scenario B:** when \( j > \ell \), \( z_i^* \) can be expressed in terms of \( \tau_i^* \) for \( i \leq \ell \) based on Case 1 using (44), where \( U_1 \triangleq \{1, \ldots, \ell\} \). Also, \( \tau_i^* \) for \( \ell < i \leq j \) can be expressed based on Case 3 as

\[
 \tau_i^* = \frac{z_i^*}{4\lambda^2} \quad \forall i \in U_3, \tag{48}
\]

where \( U_3 \triangleq \{\ell + 1, \ldots, j\} \), and \( U_4 \triangleq \{j+1, \ldots, K\} \). Consequently, \( \lambda \) can be obtained by solving C1 and C2 as:

\[
 \lambda = \frac{1}{2} \sqrt{\frac{P_M - (K - j) z_{\min}}{1 - \sum_{i \in U_4} \tau_{\max,i}}}. \tag{49}
\]

Then feasibility of the \( \ell \), and \( j \) configurations is determined by satisfying the following set of conditions:

\[
 \frac{z_{\min}}{4\lambda^2} \leq \min_{i \in U_3} \tau_{\max,i}, \quad \tau_{\min} \leq \min_{i \in U_1} \tau_{\max,i} \quad \text{and} \quad \ell \max \left( \frac{z_{\min}}{4\lambda^2}, \tau_{\min} \right) \leq 1 - \sum_{i \in U_4 \cup U_3} \tau_{\max,i} \leq \sum_{i \in U_1} \tau_{\max,i}. \tag{50}
\]
Algorithm IV: Illumination optimization

1: Input $P_M, \tau_{\min}, z_{\min}, \tau_{\max, i}, \forall i \in \{1, \ldots, K\} \backslash$
2: for $\ell = K, \ldots, 2, 1$
3: for $j = K, \ldots, 2, 1$
4: maxObj $\leftarrow$ 0, temp $\leftarrow$ 0
5: if $j \leq \ell$
6: $U_1 \leftarrow \{1, \ldots, j\}$, $U_2 \leftarrow \{j + 1, \ldots, \ell\}$, then compute $\lambda$ from (46)
7: if $\tau_{\min} \leq \frac{z_{\min}}{4\lambda^2} \leq \min_{i \in U_2} \tau_{\max, i}$ and $j \frac{z_{\min}}{4\lambda^2} \leq 1 - \sum_{i \in U_4} \tau_{\max, i} - \sum_{i \in U_1} \tau_{\max, i} - (\ell - j) \frac{z_{\min}}{4\lambda^2} \leq \sum_{i \in U_1} \tau_{\max, i}$
8: $z_i \leftarrow z_{\min} \forall i \in U_3$, $\tau_i \leftarrow \tau_{\max, i} \forall i \in U_3$
9: Find $\tau_i \forall i \in U_1$ s.t. $\frac{z_{\min}}{4\lambda^2} \leq \tau_i \leq \tau_{\max, i}$ $\forall i \in U_1$, $\sum_{i \in U_1} \tau_i = 1 - \sum_{i \in U_4 \cup U_3} \tau_{\max, i} - (\ell - j) \frac{z_{\min}}{4\lambda^2}$
10: end if
11: else
12: $U_1 \leftarrow \{1, \ldots, \ell\}$, $U_3 \leftarrow \{\ell + 1, \ldots, j\}$, then compute $\lambda$ from (49)
13: if $\frac{z_{\min}}{4\lambda^2} \leq \min_{i \in U_3} \tau_{\max, i}$, $\tau_{\min} \leq \min_{i \in U_1} \tau_{\max, i}$, and $\ell \max_{i \in U_1} \tau_{\max, i} \leq 1 - \sum_{i \in U_4 \cup U_3} \tau_{\max, i} - \sum_{i \in U_1} \tau_{\max, i}$
14: $z_i \leftarrow z_{\min} \forall i \in U_4$, $\tau_i \leftarrow \tau_{\max, i} \forall i \in U_4$
15: Find $\tau_i \forall i \in U_1$ s.t. $\max_{i \in U_1} \left(\frac{z_{\min}}{4\lambda^2}, \tau_{\min}\right) \leq \tau_i \leq \tau_{\max, i} \forall i \in U_1$, $\sum_{i \in U_1} \tau_i = 1 - \sum_{i \in U_3 \cup U_4} \tau_{\max, i}$
16: end if
17: end if
18: $z_i \leftarrow 4\lambda^2 \tau_i$ $\forall i \in U_1$, temp $\leftarrow \sum_{i=1}^{K} \sqrt{z_i \tau_i}$
19: if (temp $>$ maxObj)
20: maxObj $\leftarrow$ temp, $\tau^* \leftarrow \tau$, $z^* \leftarrow z$
21: end if
22: Solve (40) and (41) for $\nu_i \forall i$ and $o_i \forall i$
23: if $o_i \geq 0, \nu_i \geq 0$ $\forall i$
24: Terminate.
25: end if
26: end for
27: end for
28: Output: $z^*$, $\tau^*$
After asserting the configuration feasibility, $\tau_i^*$ where $i \in U_1$ are allocated any arbitrary allocation that satisfies feasibility. Then the rest of $z_i^*$ and $\tau_i^*$ variables are calculated as per the previous discussion. Finally, the solution optimality is checked by calculating the dual variables from (40) and (41).

Both scenarios are implemented in Algorithm IV, where we scan at most $(K - 1)^2$ different $\ell$ and $j$ configurations. For each configuration, the feasibility is tested and all unknown primal variables are calculated in addition to evaluating the objective function to keep track of the best feasible solution found in different configurations. Finally, the dual variables are calculated to assess optimality of the obtained solution and terminate the algorithm accordingly.

VI. SPECTRAL EFFICIENCY-ILLUMINATION TRADEOFF

In this section, we investigate the MOOP between the SE and the illumination to obtain the Pareto-frontier curve. The MOOP enables us to tune the priority level of the objective functions. Moreover, it shows the loss of SE(illumination) when illumination(SE) is maximized. The MOOP problem is formulated as

$$\text{(P3)} \quad \max_{\mathbf{z}, \mathbf{\tau}} \quad \alpha \sum_{i=1}^{K} \frac{\tau_i}{2} \ln \left(1 + \frac{\gamma_i z_i}{\tau_i}\right) + (1 - \alpha) \sum_{i=1}^{K} \sqrt{z_i \tau_i}$$

subject to

\[ C_1 - C_5. \]

(P3) is clearly convex, as the constraints represent a convex set which was discussed before, and it is a maximization problem with positive weighted sum of concave functions. We express the Lagrangian of this problem as follows:

\[ \mathcal{L} = \alpha \sum_{i=1}^{K} \frac{\tau_i}{2} \ln \left(1 + \frac{\gamma_i z_i}{\tau_i}\right) + (1 - \alpha) \sum_{i=1}^{K} \sqrt{z_i \tau_i} + \mu \left(P_M - \sum_{i=1}^{K} z_i \right) \]

\[ + \sum_{i=1}^{K} o_i \left(z_i - z_{\text{min}}\right) + \sum_{i=1}^{K} \nu_i \left(\tau_{i,\text{max}} - \tau_i\right) + \lambda \left(1 - \sum_{i=1}^{K} \tau_i\right) + \sum_{i=1}^{K} \kappa_i \left(\tau_i - \tau_{\text{min}}\right). \]

The KKT stationarity conditions of (P3) are:

\[ \frac{\partial \mathcal{L}}{\partial z_i} = \frac{\alpha}{2} \left(1 + \frac{\gamma_i z_i}{\tau_i}\right) + \frac{1 - \alpha}{2} \sqrt{\frac{1}{\tau_i}} + o_i - \mu = 0 \quad \forall i \]  

\[ \frac{\partial \mathcal{L}}{\partial \tau_i} = \frac{\alpha}{2} \left(1 - \frac{\gamma_i z_i}{\tau_i}\right) - \frac{1 - \alpha}{2} \left(1 + \frac{\gamma_i z_i}{\tau_i}\right) + \frac{1 - \alpha}{2} \sqrt{\frac{1}{\tau_i}} + \nu_i - \lambda = 0 \quad \forall i \]

in addition to $C_1 - C_5$, (19) and (18). Due to the inherent complexity in solving (51) and (52) for $z_i/\tau_i$ in terms of the Lagrange multipliers, we use the bound $\ln (1 + x) \leq \sqrt{x}$ to upper
bound the objective function of \((P3)\). We formulate \((\tilde{P}3)\) using that upper bound as a surrogate objective function. Thus, the optimal solution of \((\tilde{P}3)\) is sub-optimal for \((P3)\). However, due to similarity in behavior of the bound and the original objective function, we shall see in the simulation results in section VIII that the Pareto-frontiers obtained by optimizing the bound and the original objective function are close. The resulting surrogate optimization problem can be expressed as:

\[
(\tilde{P}3) \quad \max_{\mathbf{z}, \tau} \quad \alpha \sum_{i=1}^{K} \frac{1}{2} \sqrt{\gamma_i \tau_i z_i} + (1 - \alpha) \sum_{i=1}^{K} \sqrt{z_i \tau_i}
\]

subject to \( C1 - C5 \).

We express the Lagrangian of \((\tilde{P}3)\) as:

\[
L_{ub} = \sum_{i=1}^{K} \Gamma_i \sqrt{z_i \tau_i} + \mu \left( P_M - \sum_{i=1}^{K} z_i \right) + \sum_{i=1}^{K} \alpha_i (z_i - z_{\text{min}}) + \sum_{i=1}^{K} \nu_i (\tau_i, \text{max} - \tau_i)
\]

\[+ \lambda \left( 1 - \sum_{i=1}^{K} \tau_i \right) + \sum_{i=1}^{K} \kappa_i (\tau_i - \tau_{\text{min}}), \]

where \(\Gamma_i = \frac{\alpha}{2} \sqrt{\gamma_i} + 1 - \alpha\).

The KKT conditions of \((\tilde{P}3)\) are:

\[
\frac{\partial L_{ub}}{\partial z_i} = 0 \quad \Rightarrow \quad \frac{z_i^*}{\tau_i^*} = \frac{\Gamma_i^2}{4 (\mu - \alpha_i)^2} \quad \forall i
\]

\[
\frac{\partial L_{ub}}{\partial \tau_i} = 0 \quad \Rightarrow \quad \frac{\tau_i^*}{z_i^*} = \frac{\Gamma_i^2}{4 (\nu_i + \lambda - \kappa_i)^2} \quad \forall i
\]

in addition to \(C1 - C5, (19)\) and \((18)\).

As mentioned in the two former sections, we avoid the exponential complexity of searching over all possible combinations of the inequalities Lagrange multipliers by restricting our search space to solutions having special structures. For this problem we look for solutions of two possible forms:

(i) \(\mathbf{z}^* = [z_1^*, \ldots, z_j^*, z_{\text{min}}, \ldots, z_{\text{min}}]^T\), \(\mathbf{\tau}^* = [\tau_{\text{max},1}, \ldots, \tau_{\text{max},f-1}, \tau_f^*, \ldots, \tau_{\text{min}}, \ldots, \tau_{\text{min}}]^T\).

(ii) \(\mathbf{z}^* = [z_1^*, \ldots, z_j^*, z_{\text{min}}, \ldots, z_{\text{min}}]^T\), \(\mathbf{\tau}^* = [\tau_{\text{max},1}, \ldots, \tau_{\text{max},f-1}, \tau_f^*, \ldots, \tau_{\text{max},l+1}, \ldots, \tau_{\text{max},K}]^T\).

We choose the previous structures to limit the complexity of solving the KKT system to \(O(K^3)\).

The structure (i) is chosen inspired by the SE maximization problem solution, while the structure (ii) is chosen inspired by the superposition of the structures of both the SE and the illumination maximization problems. In contrast to SE maximization problem, one user at most can fall into Case 1 in Table III otherwise the KKT system will have no solution. However, similar to the
TABLE III: Inequality constraints multipliers configurations for (\(P_3\)). The optimal \((\tau_i, z_i)\) pair is indicated below each case.

<table>
<thead>
<tr>
<th>(\nu_i)</th>
<th>(\kappa_i)</th>
<th>(o_i = 0)</th>
<th>(o_i \neq 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\nu_i = 0)</td>
<td>(\kappa_i = 0)</td>
<td>Case 1 ((\tau_i^<em>, z_i^</em>)) (z_i^* = \frac{\tau_i^2}{4\mu^2}), (z_i^* = \frac{\tau_i^2}{4\lambda^2})</td>
<td>Case 4 ((\tau_i^*, z_{\min})) (z_{\min} = \frac{\tau_i^2}{4(\mu-\kappa_i)^2}), (z_{\min} = \frac{\tau_i^2}{4\lambda^2})</td>
</tr>
</tbody>
</table>
| \(\nu_i 
eq 0\) | \(\kappa_i 
eq 0\) | Case 3 \((\tau_{\min, i}, z_i^*)\) \(z_i^* = \frac{\tau_i^2}{4\mu^2}\), \(z_i^* = \frac{\tau_i^2}{4(\lambda-\kappa_i)^2}\) | Case 5 \((\tau_{\min}, z_{\min})\) \(z_{\min} = \frac{\tau_i^2}{4(\mu-\kappa_i)^2}\), \(z_{\min} = \frac{\tau_i^2}{4(\lambda-\kappa_i)^2}\) |
| \(\nu_i 
eq 0\) | \(\kappa_i = 0\) | Case 2 \((\tau_{\max, i}, z_i^*)\) \(z_i^* = \frac{\gamma_i^2}{4\mu^2}\), \(z_i^* = \frac{\gamma_i^2}{4(\lambda-\kappa_i)^2}\) | Case 6 \((\tau_{\max, i}, z_{\min})\) \(z_{\min} = \frac{\gamma_i^2}{4(\mu-\kappa_i)^2}\), \(z_{\min} = \frac{\gamma_i^2}{4(\lambda-\kappa_i)^2}\) |

For the SE problem, we have three possible solution structures: (i) \(N_\tau > 1, j = f\), (ii) \(N_\tau > 1, j < f\), and (iii) \(N_\tau = 1\).

(i) For \(N_\tau > 1, j = f\), we calculate \(\lambda\) as

\[
\lambda = \frac{1}{2} \sqrt{\left( P_M - \left( K - j - \frac{\sum_{i \in U_2} \tau_i^2}{\Gamma_j^2} \right) \frac{z_{\min}}{\Gamma_j^2} \right)}
\]

where \(S\) is a binary variable that is set to either 1 or 0 to represent the first or the second solution structure, respectively. Then \(\mu\) is calculated from:

\[
\mu = \frac{\gamma_j^2}{4\lambda}
\]

Finally all primal variables can be obtained as follows:

\[
z_i^* = \tau_{\max, i} \frac{\gamma_i^2}{4\mu^2} \quad \forall i \in U_2, \quad U_2 \triangleq \{1, \ldots, j - 1\}
\]

\[
\tau_i^* = z_{\min} \frac{\gamma_i^2}{4\lambda^2} \quad \forall i \in U_4, \quad U_4 \triangleq \{j + 1, \ldots, l\}
\]

\[
z_j^* = P_M - z_{\min} (K - j) - \sum_{i \in U_2} z_i^*
\]

\[
\tau_j^* = 1 - \tau_{\min} (K - f) S + (1 - S) \sum_{i = l + 1}^K \tau_{\max, i} - \sum_{i \in U_4} \tau_{\max, j},
\]

(ii) For \(N_\tau > 1, j < f\), we get \(\lambda\), from

\[
\lambda = \frac{1}{2} \sqrt{\frac{\gamma_j^2 \sum_{i \in U_4} \frac{\tau_i^2}{\Gamma_i^2}}{1 - \tau_{\min} (K - f) S + (1 - S) \sum_{i = l + 1}^K \tau_{\max, i} - \sum_{i \in U_2} \tau_{\max, i}}}
\]

where \(U_2 \triangleq \{1, \ldots, j\}\), and \(U_4 \triangleq \{j + 1, \ldots, l\}\), and we calculate \(\mu\) from

\[
\mu = \frac{1}{2} \sqrt{\frac{\sum_{i \in U_2} \frac{\tau_i^2}{\Gamma_i^2}}{P_M - z_{\min} (K - j)}}
\]
TABLE IV: Default Simulation Parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_0$</td>
<td>$10^{-21}$ W/Hz</td>
</tr>
<tr>
<td>$B_v$</td>
<td>20 MHz</td>
</tr>
<tr>
<td>$H$</td>
<td>6.75 m</td>
</tr>
<tr>
<td>$P_{cr}$</td>
<td>200 mW</td>
</tr>
<tr>
<td>$\phi_A$</td>
<td>60°</td>
</tr>
<tr>
<td>$P_M$</td>
<td>1000 W</td>
</tr>
<tr>
<td>$K$</td>
<td>30</td>
</tr>
<tr>
<td>$R_{th}$</td>
<td>10 Kbps</td>
</tr>
<tr>
<td>$\tau_{min}$</td>
<td>$7.14 \times 10^{-4}$ s</td>
</tr>
<tr>
<td>$A_{PD}$</td>
<td>1 cm$^2$</td>
</tr>
<tr>
<td>$R_{PD}$</td>
<td>3 A/W</td>
</tr>
<tr>
<td>$I_o$</td>
<td>$1 \times 10^{-12}$ A</td>
</tr>
<tr>
<td>$\psi_A$</td>
<td>80°</td>
</tr>
</tbody>
</table>

respectively, then $z_i^* \forall i \in U_2, \tau_i^* \forall i \in U_4$ are calculated from (58), (59) respectively.

(iii) For $N_r = 1$, $\mu$ is calculated from

$$\mu = \frac{1}{2} \sqrt{\frac{\sum_{i=1}^{j} \Gamma_i^2}{P_M - z_{\min}(K-j)}}.$$  

Then $\tau_f^* = 1 - \sum_{i=1}^{f-1} \tau_{max,i} - \tau_{min}(K-l) S + (1-S) \sum_{i=l+1}^{K} \tau_{max,i}$, and $z_i^* = \frac{\Gamma_i^2 r_i^*}{4\mu^2} \forall i \leq j$.

In Algorithm I with ($A = 0$), we first assume that the first solution structure is optimal, search through different inequalities multipliers configurations (i)–(iii) in a very similar way to Algorithm I. Then for each configuration, we calculate the primal variables as per the previous discussion. We keep track of the best found feasible configuration and calculate the dual variables and terminate in case optimality is reached (all KKT conditions are satisfied). In case the algorithm didn’t terminate after scanning all possibilities for the first solution structure, we repeat the same steps whilst considering the second solution structure.

VII. SIMULATION RESULTS

The simulation setup consists of a VLC transmitter similar to the scenario shown in Fig. 1 with a 7.6 m clearance from ground. The transmitter coverage is represented by a circle with radius $R_{\text{outer}} = 7.6 \tan(\psi_A)$. We assume that users are distributed over a circular disc with inner and outer radii $R_{\text{inner}}$ and $R_{\text{outer}}$, respectively. All the users’ receivers are assumed to have horizontal orientation as the transmitter. The average SE and illumination results are calculated based on 1000 realizations of users placed at random locations according to a uniform distribution. Unless otherwise stated, we use the simulation parameters listed in Table IV. In the conducted simulations, infeasible realizations have zero contribution to the average SE and illumination performance metrics.

A. SE Optimization Simulations

In this section, we study the average optimized SE performance of the considered VLC system for different proposed algorithms, i.e., I, II, III. As a benchmark, we use the interior point method
algorithm that gives optimal solution for the considered problem due to its convexity. In the following simulation example, we study the effect of changing the service area of the proposed system on the SE performance. To this end, we vary $R_{\text{inner}}$ and keep $R_{\text{outer}}$ constant for different values of $R_{\text{th}}$ and $P_M$ as shown in Fig. 2 and Fig. 3, respectively. It can be noticed that as $R_{\text{inner}}$ increases, the SE performance degrades because the users get far from the transmitter, which worsens the channels gain.

As for the relative performance of our proposed algorithms and the interior point solution, we observe that Algorithm I proves its efficiency and achieves similar performance to the interior point method for different ranges of $P_M$ and $R_{\text{th}}$. As for Algorithm II, it has a negligible gap with Algorithm I for small and large values of $P_M$, while it has a small gap for the mid-range of $P_M$ as can be noticed from Fig. 3. Similar behavior is observed with the increase of $R_{\text{th}}$ in Fig. 2. On the other hand, the performance gap between Algorithm I and algorithm III is monotonically increasing with $P_M$ as shown in Fig. 3, which is due to the uniform power allocation of algorithm III. Similar observations hold with the increase of $R_{\text{th}}$ in Fig. 2. It should be noted that performance of algorithm III does not change for $R_{\text{th}} = \{5, 18, 19\}$ because $R_{\text{th}}$ affects the power allocation (which is constant for algorithm III) and the problem feasibility which is identical for $R_{\text{th}} = \{5, 18, 19\}$. Albeit, it can be seen clearly in Fig. 3 that algorithm III performance gets enhanced as $P_M$ increases for the same previously mentioned reasons.
B. Illumination optimization simulations

In this section, we study the performance of the proposed illumination optimization algorithm (Algorithm IV) and the optimal performance (using interior point method) versus different system parameters. In the first simulation example, we study the illumination performance versus $R_{th}$ for different energy harvesting requirements ($\beta$) as shown in Fig. 4. We observe that the performance of Algorithm IV matches the interior point method performance. In terms of illumination performance, we observe that when the energy harvesting requirement gets very loose, the average obtainable illumination will not vary with $R_{th}$. It is worth to mention that decreasing the illumination performance is due to outages. Since changing $R_{th}$ will not affect the feasibility of the solution $z_i^* = P_M r_i^*$, thus the decreases in illumination performance for $\beta = 0.005$, $R_{th} \geq 16$ Kbps are due to outages.

Next, we study the illumination performance versus $P_M$ for different $\beta$ in Fig. 5. As $P_M$ increases, the feasibility region gets larger, thus, the average illumination increases as shown in Fig. 5. The presented result shows also that as $\beta$ increases, the illumination gains of increasing $P_M$ become more significant. It is worthy emphasizing that Algorithm IV performance matches the interior point method performance as shown in Fig. 5 similar to the previous simulation example.
C. SE–Illumination optimization simulations

In this section, we study the tradeoff relationship between the SE and the illumination using Algorithm I and compare it with the interior point algorithm solution of the original problem (P3) and the upper bounded problem (˜P3). The first set of SE-illumination Pareto fronts shows the tradeoff relationship for different β in Fig. 6. As β increases, illumination intensity and SE performance degrades, because increasing β results in a tighter feasibility region and higher chances for infeasible realizations.

Next, the average SE-illumination Pareto front obtained by the three aforementioned methods are plotted for different values of PM in Fig. 7. The simulation results show that as PM increases, the achievable average SE-illumination trade-off is improved as the increased power budget
makes higher SE and illumination intensities attainable. Then, the Pareto front is studied with the change of $R_{th}$ for $\beta = 0.005$ as shown in Fig. 8. As $R_{th}$ increases, the Pareto front curve is degraded at the SE side without changing the maximum illumination. Changing $R_{th}$ does not affect the illumination performance function if the energy harvesting requirement is tolerant enough. All the considered simulations have shown that Algorithm I performance matches the performance of optimal solution to the surrogate optimization problem ($\tilde{P}_3$) and it is so close to the optimal solution of the original problem ($P_3$).

Studying the tradeoff strength is essential in providing good guidelines for system designers. One way is to investigate the skewness of the Pareto front (the ratio of SE loss to illumination loss) change with different parameters ($R_{th}$, $\beta$, and $P_M$). It is observed that as the constraints get looser ($\beta$ or $R_{th}$ decreases or $P_M$ increases) the SE losses due to illumination optimization becomes more significant than the illumination losses due to optimizing SE. For this purpose, we study the SE loss percentage defined as the difference between maximum achievable SE and the highest attainable SE when illumination is maximized, divided by maximum SE. In addition to that, we study the illumination loss that has a similar definition to the SE loss. In Fig. 9, we observe that as $P_M$ increases, both the SE and illumination loss percentages exhibit different trend either a decreasing, unimodal, or increasing depending on the value of $\beta$. It is beneficial to monitor when the loss becomes significant to consider it in the design process. Fig. 10 shows the SE and illumination loss versus the $R_{th}$ for different $\beta$. It is worthy to note that
the loss is significant in both SE and illumination for negligible harvesting and high minimum rate requirements. However, loss decreases with the increase of $\beta$ and become significant for the illumination at high $R_{th}$.

VIII. CONCLUSION

In this work, we considered the resource allocation problem for outdoor VLC system employing dynamic TDMA for outdoor scenario where energy harvesting is enabled. We showed that the proposed system has a tradeoff relationship between the communication and optical services. MOOP becomes a powerful optimization framework to deal with such scenarios, and hence it is important to characterize the Pareto front of the SE-illumination region. To this end, we solved the marginal optimization problems, maximizing the SE and maximizing the illumination. Then, inspired by the proposed solution, we suggested an algorithm that obtain the Pareto front of the SE-illumination region. Numerical results showed a close match between our algorithms and the optimal solution.

APPENDIX A

Define $f(\tau; z, \gamma) = \tau \ln (1 + \gamma z)$ with respect to $\tau$ for $0 \leq \tau \leq 1$ and $g(\tau; z, \gamma)$ as $\frac{\partial f}{\partial \tau} = \ln (1 + \gamma z) - \frac{\gamma z}{1 + \gamma z}$. By calculating the first derivative of $g(\tau; z, \gamma)$ with respect to $\tau$ we get,

$$\frac{\partial g}{\partial \tau} = -\frac{z^2 \gamma}{(1 + \gamma z)^2} \leq 0.$$  \hspace{1cm} (65)

Since $g(\tau; z, \gamma)$ is monotonically decreasing in $\tau$ in the interval $0 \leq \tau \leq 1$, then, $\frac{\partial f}{\partial \tau} = \ln (1 + \frac{\gamma z}{\tau}) \leq \ln (1 + \gamma z) - \frac{\gamma z}{1 + \gamma z}$.

Now, let us consider the function $P(x) = \ln (1 + x) - \frac{x}{1+x} = \ln (1 + x) + \frac{1}{x+1} - 1$, where $x \in [0, \infty)$. This satisfies $P(0) = 0$, $\lim_{x \to \infty} P(x) = \infty$, $P'(x^*) = \frac{x^*}{(x^*+1)^2} = 0 \Rightarrow x^* = 0$ and $P''(x^*) = \frac{1-x^*}{(x^*+1)^3} \Rightarrow P''(0) = 1 > 0$, which certifies that $x = 0$ is a local minimum for $P(x)$. Consequently, the global minimum of $P(x)$ for $x \in [0, \infty)$ is zero. Thus, $\ln (1 + \gamma z) - \frac{\gamma z}{1 + \gamma z} \geq 0$, which implies that $g(\tau; z, \gamma) \geq 0$ for $0 \leq \tau \leq 1$, which proves the positive monotonicity of $f(\tau; z, \gamma)$ for $0 \leq \tau \leq 1$ and $z \geq 0$.

APPENDIX B

For the considered configuration to be feasible, $\tau_j \geq \tau_{\text{min}} \geq 0$ must hold, which implies that:

$$\left(1 - \sum_{i=1}^{f-1} \tau_{\text{max},i} - (K - \ell) \tau_{\text{min}} - \sum_{i \in U_4} g_i(\lambda)\right) \geq \tau_{\text{min}} \geq 0.$$  \hspace{1cm} (66)
By studying the monotonicity of \( g_i(\lambda) \) we get:

\[
g'_i(\lambda) = -z_{\min} \gamma_i \left( W_0 \left( -e^{-(\lambda+1)} \right) + 1 \right)^2 e^{-(\lambda+1)} \leq 0. \tag{67}
\]

For the two previous inequalities to hold \( \lambda \leq \lambda_{\max} \) must be satisfied, where \( \sum_{i \in U_4} g_i(\lambda_{\max}) = 1 - \sum_{i=1}^{f-1} \tau_{\max,i} - (K - \ell) \tau_{\min} - \tau_{\min} \).

To check the monotonicity of \( G(\lambda) \), we differentiate (30) with respect to \( \lambda \):

\[
\frac{dG}{d\lambda} = \left( 1 - \sum_{i=1}^{f-1} \tau_{\max,i} - (K - \ell) \tau_{\min} - \sum_{i \in U_4} g_i(\lambda) \right) e^{-\lambda-1} W'_0 \left( -e^{-(\lambda+1)} \right) \frac{\gamma_j}{\gamma_j} \left( W_0 \left( -e^{-(\lambda+1)} \right) \right)^2 \geq 0 \text{ for } \lambda \leq \lambda_{\max} \\
+ g'_i(\lambda) \left( \frac{1}{\gamma_j} W'_0 \left( -e^{-(\lambda+1)} \right) + \frac{1}{\gamma_j} \right) - \gamma_j \sum_{i \in U_2} f'_i \left( -\gamma_j W_0 \left( -e^{-(\lambda+1)} \right) \right) W'_0 \left( -e^{-(\lambda+1)} \right) e^{-(\lambda+1)} \leq 0 \text{ for } \lambda \leq \lambda_{\max} \tag{68}
\]

Thus, by observing the signs of the factors in (68), we conclude that \( \frac{dG}{d\lambda} \geq 0 \) for \( \lambda \leq \lambda_{\max} \).

REFERENCES


