MIMO Radar Transceiver Design for High Signal-to-Interference-Plus-Noise Ratio

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ABSTRACT

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Multiple-input multiple-output (MIMO) radar employs orthogonal or partially correlated transmit signals to achieve performance benefits over its phased-array counterpart. It has been shown that MIMO radar can achieve greater spatial resolution, improved signal-to-noise ratio (SNR) and target localization, and greater clutter resolution using space-time adaptive processing (STAP). This thesis explores various methods to improve the signal-to-interference-plus-noise ratio (SINR) via transmit and receive beamforming.

In MIMO radar settings, it is often desirable to transmit power only to a given location or set of locations defined by a beampattern. Current methods involve a two-step process of designing the transmit covariance matrix $\mathbf{R}$ via iterative solutions and then using $\mathbf{R}$ to generate waveforms that fulfill practical constraints such as having a constant-envelope or drawing from a finite alphabet. In this document, a closed-form method to design $\mathbf{R}$ is proposed that utilizes the discrete Fourier transform (DFT) coefficients and Toeplitz matrices. The resulting covariance matrix fulfills the practical constraints such as positive semidefiniteness and the uniform elemental power constraint and provides performance similar to that of iterative methods, which require a much greater computation time. Next, a transmit architecture is presented.
that exploits the orthogonality of frequencies at discrete DFT values to transmit a sum of orthogonal signals from each antenna. The resulting waveforms provide a lower mean-square error than current methods at a much lower computational cost, and a simulated detection scenario demonstrates the performance advantages achieved.

It is also desirable to receive signal power only from a given set of directions defined by a beampattern. In a later chapter of this document, the problem of receive beampattern matching is formulated and three solutions to this problem are demonstrated. We show that partitioning the received data vector into subvectors and then multiplying each subvector with its corresponding weight vector can improve performance and reduce the length of the data vector. Simulation results show that all methods are capable of matching a desired beampattern. Signal-to-interference-plus-noise ratio (SINR) calculations demonstrate a significant improvement over the unaltered MIMO case.
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LIST OF ABBREVIATIONS

APES  Amplitude-and-Phase Estimation
BPSK  Binary Phase-Shift Keying
CE    Constant Envelope
DFT   Discrete Fourier Transform
FFT   Fast Fourier Transform
GN    Gauss-Newton
LS    Least Squares
LSE   Least Squares Error
MIMO  Multiple-Input Multiple-Output
MSE   Mean Square Error
MVDR  Minimum Variance Distortionless Response
PAPR  Peak-to-Average Power Ratio
QPSK  Quadrature Phase-Shift Keying
RCS   Radar Cross Section
ROI   Region of Interest
SAR   Synthetic Aperture Radar
SINR  Signal-to-Interference-Plus-Noise Ratio
SNR   Signal-to-Noise Ratio
SQP  Semidefinite Quadratic Program
STAP  Space-Time Adaptive Processing
ULA  Uniform Linear Array
LIST OF SYMBOLS

\( N_R \) \quad \text{Number of receive antennas in array}
\( N_T \) \quad \text{Number of transmit antennas in array}
\( P_d(\theta_l) \) \quad \text{Power received from a target at location } \theta_l
\( P_r(\theta) \) \quad \text{Power received from a target at location } \theta
\( P_t(\theta) \) \quad \text{Power transmitted to a target at location } \theta
\( \beta(\theta) \) \quad \text{Reflection coefficient of a target at location } \theta
\( \lambda \) \quad \text{Wavelength of transmitted signal}
\( \mathbf{R} \) \quad \text{Covariance matrix of transmitted MIMO signals}
\( \mathbf{a}_R(\theta) \) \quad \text{Receive steering vector at location } \theta
\( \mathbf{a}_T(\theta) \) \quad \text{Transmit steering vector at location } \theta
\( \mathbf{x}(n) \) \quad \text{Transmitted baseband signal vector at time } n
\( \theta \) \quad \text{Location angle; degrees or radians specified within text}
\( d \) \quad \text{Interelement spacing}
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Chapter 1

Introduction

1.1 A Brief Review of Radar

Radar systems have been utilized throughout the previous decades for a variety of applications, from military surveillance to the measurement of environmental features [1]. Typical applications perform tasks such as detection and tracking of targets and the estimation of their parameters, such as range, velocity, and reflection coefficient. More sophisticated tasks, such as Synthetic Aperture Radar (SAR) imaging, can be performed to provide images of both earth and space. As advances in technology have reduced the cost of antennas and processors, radar systems have been employed in consumer motor vehicles to warn of eminent crashes, avoid collisions, and provide adaptive cruise control for safer driving in inclement weather [2, 3, 4]. Specific details on these applications can be found in the text [1] and are beyond the scope of this document.

1.1.1 Phased-Array Signal Model

In this section and throughout this document, it is assumed that the transmitted signal is narrowband and that the target receiving the signal lies in the far field and is stationary (i.e., no Doppler shift is present). Until recently, radar arrays have
utilized what is known as the phased-array architecture [1], in which each antenna transmits a scaled and phase-shifted version of the same signal. For a Uniform Linear Array (ULA) array with $N_T$ transmit antennas having interelement spacing $d$ and transmission wavelength $\lambda$, the signal received by a target at location $\theta$ is

$$r(n; \theta) = a_T^H(\theta)x(n),$$  

(1.1)

where $a_T(\theta)$ is the transmit steering vector and is defined as

$$a_T(\theta) = \begin{bmatrix} 1 & e^{-j \frac{2\pi d}{\lambda} \sin(\theta)} & \ldots & e^{-j \frac{2\pi d}{\lambda} \sin(\theta)(N_T - 1)} \end{bmatrix}^T,$$  

(1.2)

and $x(n)$ is the vector of baseband signals transmitted from each antenna

$$x(n) = \begin{bmatrix} x_1(n) & x_2(n) & \ldots & x_{N_T}(n) \end{bmatrix}^T.$$

(1.3)

Note that there is also a term representing the carrier frequency in the actual transmitted signal. However, this term is irrelevant in the following work, as it disappears either under the expectation or after matched-filtering. Hence, we work only with the baseband signal representations. The steering vector is a result of the difference in path lengths between signals transmitted from different antennas. An illustration of its origin can be seen in Fig. 1.1.

As mentioned previously, $x(n)$ may consist of amplitude and phase-shifted versions of the same signal. Let the vector of phase-shifts be $w$ and consider the case of no scaling. Then the signal received by the target becomes

$$r_{PA}(n; \theta) = a_T^H(\theta)wx(n).$$  

(1.4)

In order to steer the maximum energy of the beam toward a target at angle $\theta_t$, one
can set $\mathbf{w} = \mathbf{a}_T(\theta_t)$, resulting in the received signal

$$
\begin{align*}
    r_{PA}(n; \theta) &= a_H^T(\theta)\mathbf{a}_T(\theta)\mathbf{x}(n) \\
    &= N_T\mathbf{x}(n), \quad \theta = \theta_t. \quad (1.5)
\end{align*}
$$

The transmitted power to the location $\theta$, $P_t(\theta)$, which is referred to in this document as the transmit beampattern, can be written as

$$
\begin{align*}
    P_t(\theta) &= E\{a_H^T(\theta)\mathbf{a}_T(\theta)\mathbf{x}(n)x^*(n)a_H^T(\theta_t)\mathbf{a}_T(\theta)\} \\
    &= E\{x(n)x^*(n)\} \mathbf{a}_T^H(\theta)\mathbf{a}_T(\theta) \mathbf{a}_T^H(\theta_t)\mathbf{a}_T(\theta) \\
    &= \mathbf{a}_T^H(\theta)\mathbf{a}_{Tt}(\theta)\mathbf{a}_T^H(\theta_t)\mathbf{a}_T(\theta) \\
    &= |\mathbf{a}_T^H(\theta)\mathbf{a}_T(\theta_t)|^2 \\
    &= \left|\frac{1 - e^{jN_T\sin(\theta - \theta_t)}}{1 - e^{j\psi\sin(\theta - \theta_t)}}\right|^2. \quad (1.6)
\end{align*}
$$

When the transmit beam is steered toward the target of interest, it is easily seen that $P_t(\theta_t) = N_T^2$. This is known as the coherent processing gain and is a significant advantage of the phased-array system. A diagram of the phased-array transmit architecture can be seen in Fig. 1.2, where $\psi = \frac{2\pi d}{\lambda}\sin(\theta)$. The signal at the receive array encounters a similar phase shift due to the antenna
Figure 1.2: Illustration of the phased-array transmit architecture, where $\psi = \frac{2\pi d}{\lambda} \sin(\theta)$.

locations, described by the receive steering vector, $a_R(\theta)$, with length equal to the number of receive antennas, $N_R$,

$$
a_R(\theta) = \left[ e^{-j2\pi d\sin(\theta)} \ldots e^{-j2(N_R-1)d\sin(\theta)} \right]^T, \tag{1.7}
$$

resulting in the received signal vector

$$
y(n; \theta) = \beta(\theta) a_R(\theta) a^H_T(\theta) a_T(\theta_t)x(n), \tag{1.8}
$$

where $\beta(\theta)$ is the complex reflection coefficient of the target located at angle $\theta$ and is the magnitude squared of the Radar Cross Section (RCS). In the typical receiver, the received signals are matched filtered to obtain

$$
y(\theta) = \beta(\theta) a_R(\theta) a^H_T(\theta) a_T(\theta_t). \tag{1.9}
$$

In many radar applications, the goal is to estimate $\beta(\theta_t)$ in the presence of interferers and noise. Consider the case in which there exist a single target of interest located at $\theta_t$, $K$ interfering targets at $\{\theta_k\}_{k=1}^K$, and the length $N_R$ noise vector $v$ (usually
circularly-symmetric white Gaussian noise). The received signal is then

\[ y = \beta(\theta_t)N_T a_R(\theta_t) + \sum_{k=1}^{K} \beta(\theta_k) a_R(\theta_k) a_T^H(\theta_k) a_T(\theta_t) + \mathbf{v}. \]  

(1.10)

Given this model, the reflection coefficient can be estimated using methods such as the Minimum Variance Distortionless Response (MVDR) or Capon beamformer. However, in the case of multiple targets of interest, the Capon and other data-dependent estimation methods fail [5]. This and other shortcomings in phased-array radar can be overcome by transmitting multiple orthogonal or partially correlated signals, an architecture which is known as Multiple-Input Multiple-Output (MIMO) radar.

1.2 Introduction to MIMO Radar

The concept of MIMO radar has been introduced in recent years as a topic of interest, receiving both skepticism and praise [6, 7, 8]. Critics of MIMO radar cite the loss of coherent processing gain obtained in phased-array systems, which results in a decrease in Signal-to-Noise Ratio (SNR) by a factor of \( N_T \), as well as a decrease in the maximum useful area of range-Doppler space [7]. A MIMO system transmits \( N_T \) uncorrelated or partially correlated signals, which can be exploited at both the transmit and receive ends to obtain performance advantages [9]. In the MIMO radar problem, antennas are considered to be either widely separated or co-located.

For widely separated antennas, the spatial diversity of a target can be captured, resulting in improved SNR and high resolution target localization [10, 11]. This setup is often referred to as statistical MIMO and shares numerous similarities with MIMO communications [10]. In this case, the RCS resulting from each transmitted signal can be treated as an independent random variable, and by combining the received information detection performance can be improved.

A system utilizing co-located antennas achieves improved spatial resolution and
what is known as the virtual steering vector [12, 8], which results in greater parameter identifiability [12] and can be utilized to obtain greater clutter resolution in Space-Time Adaptive Processing (STAP) [13]. In [12], it is shown that the number of identifiable targets is \( N_T \) times the maximum achievable by a phased-array radar. Moreover, this setup enables the use of a variety of receive beamformers, from classical Capon [5] to robust versions [14, 15]. These methods allow for improved parameter identifiability, even in the case of numerous targets of interest.

One final area of interest for MIMO radar, and radar systems in general, is that of transmit signal design. In such a case, the user wishes to transmit signals such that targets in a certain Region of Interest (ROI) are illuminated, while targets outside the ROI are suppressed. This is the topic of Chapter 2, where a thorough introduction to this subject can be found.

### 1.2.1 MIMO Signal Model

In this section, the MIMO signal model transmitting orthogonal signals is described. Consider the vector of \( N_T \) orthogonal signals

\[
x(n) = [x_1(n) \ x_2(n) \ .... \ x_{N_T}(n)]^T.
\]  

(1.11)

The signal received by a target at angle \( \theta \) is then

\[
r_{MIMO}(n; \theta) = a_T^H(\theta)x(n).
\]  

(1.12)
In this case, it is easily seen that the transmit beampattern is uniform over all $\theta$

$$P_t(\theta) = E\{a_T^H(\theta)x(n)x^H(n)a_T(\theta)\}$$
$$= a_T^H(\theta)E\{x(n)x^H(n)\}a_T(\theta)$$
$$= a_T^H(\theta)Ia_T(\theta)$$
$$= N_T,$$  \hspace{1cm} (1.13)

where $I$ denotes the identity matrix of the corresponding dimension and we have made use of the fact that the transmitted signals are orthogonal. Note that the power at a desired $\theta_t$ is reduced by a factor of $N_T$ as compared to the phased-array system. However, in the MIMO case all directions are illuminated, and thus multiple targets can be reliably detected without the need for scanning.

The signal at the receiver due to the target at location $\theta$ is

$$y(n; \theta) = a_R(\theta)a_T^H(\theta)x(n).$$  \hspace{1cm} (1.14)

Applying a matched filter as in section 1.1.1, the virtual steering vector [13] can be obtained

$$y(\theta) = a_R(\theta) \otimes a_T(\theta),$$  \hspace{1cm} (1.15)

where $\otimes$ denotes the Kronecker product. This virtual steering vector provides the extra degrees of freedom that allow for enhanced resolution using MIMO radar.

### 1.3 Outline of Thesis

This thesis covers the design of MIMO radar transmit and receivers (transceivers) with the goal of improving the Signal-to-Interference-Plus-Noise Ratio (SINR). Chapter 2 describes one method of improving the SINR by designing the transmit waveforms
such that power is transmitted only to a certain region (or set of regions) of space. This is referred to as transmit beamforming. In Chapter 3, several methods for beamforming at the receive end are demonstrated. These methods allow the user to receive signal power only from a desired region (or set of regions) of space. Conclusions and future work are presented in Chapter 4. The major results of each chapter are briefly explained in this section.

1.3.1 Transmit Beampattern Design

Transmit beampattern design has received a great deal of attention in the MIMO community [16, 17, 18, 19, 20, 21, 22, 23, 24]. Transmit beamforming is typically performed by first designing the signal covariance matrix $\mathbf{R}$, and then designing actual transmit signals that match the given covariance matrix as closely as possible while fulfilling practical constraints. The methods presented in [17, 18, 19, 20, 21] rely on iterative methods of solution to find the covariance matrix. In Chapter 2 we propose a closed-form solution of covariance matrix design, which is derived from the Discrete Fourier Transform (DFT) coefficients of a frequency-domain window. The resulting performance approaches that achieved by iterative methods at a greatly reduced computational complexity. However, for a small number of transmit antennas the number of achievable beampatterns is limited. We also demonstrate a radar architecture that can be used to transmit signals to the desired region without first designing $\mathbf{R}$. This method results in a significant reduction in computational complexity and provides improved performance when compared to existing methods.

1.3.2 Receive Beampattern Matching

Receive beampattern design is a classical problem with numerous solutions, many of which are described in detail in [25]. Due to the virtual steering vector resulting from a MIMO setup, more degrees of freedom exist on the receive side and therefore im-
proved performance can be achieved. Chapter 4 outlines a series of methods designed to match an arbitrary beampattern at the receive side, many of which are natural extensions to the transmit beampattern problem. However, although the generation of the receive vector is unique to MIMO radar, once the data arrives at the receiver, the problem does not differ from classical radar/sonar problems investigated in the past. For this reason, the classical methods described in [25] can be applied directly. Unfortunately, the proposed methods do not provide any performance benefits over existing methods.
Chapter 2

Transmit Beampattern Design

2.1 Introduction

In the traditional phased-array setup, each transmit antenna transmits a phase-shifted version of the same baseband waveform. In this case, the probing signals are fully correlated and the transmit covariance matrix is that of a rank-1 beamformer, resulting in a single beam focused at the origin. A common MIMO setup is that of orthogonal or omnidirectional signaling, in which the antennas transmit mutually orthogonal waveforms. Here the transmit covariance matrix is equal to the identity matrix, and equal power is transmitted in all directions. The topic of transmit beampattern design lies between these two extremes and is the subject of this chapter.

In the transmit beampattern design problem, the user wishes to transmit power exclusively to one or more prespecified ROIs. Previous work on beampattern design relies largely on iterative methods of solution [17, 18, 19, 20, 21]. In [18], the authors present a closed-form solution based on a least-squares cost function. However, the resulting covariance matrix is not positive semidefinite and therefore requires the use of eigenvalue or singular value decomposition. The solution also fails to fulfill the uniform elemental power constraint (i.e., it does not have constant values along the main diagonal). Another closed-form solution, which does fulfill these constraints, was presented in [16]. However, this method shows significant performance degradation.
compared with iterative methods, especially in the case of wide ROIs, and is only capable of transmitting power to a single ROI. One further method of closed-form beampattern design was presented as an initial guess in [20]. This method involves a summation of phased-array beams, but has the drawbacks of slow roll-off and high sidelobe levels. Efficient numerical solutions have been presented in [17, 19, 20], though for large array sizes these methods require a nontrivial amount of calculation time.

Once the covariance matrix has been designed, the next step is to design the actual waveforms to be transmitted that have the desired cross-correlations [22, 23, 24]. In [22], the authors extend the work of [17] to generate waveforms that fulfill realistic constraints such as the Constant Envelope (CE) or low Peak-to-Average Power Ratio (PAPR). However, the designed signals come from an infinite alphabet, and the method requires the use of an iterative solution. In [23], a mapping of Gaussian random variables into Binary Phase-Shift Keying (BPSK) and Quadrature Phase-Shift Keying (QPSK) CE waveforms is described. However, this method does not guarantee a positive semidefinite covariance matrix, and thus a second algorithm is proposed to generate the best possible BPSK waveforms for the given Gaussian random variables to generate BPSK waveforms. The work in [24] extends this method to QPSK signals. In both cases, iterative methods are again required, and the resulting beampatterns suffer from high sidelobe levels.

This chapter describes two contributions within the topic of transmit beampattern design. First, we present a novel closed-form method to synthesize the covariance matrix for the desired beampattern. The method relies on the simple procedure of choosing DFT coefficients to have nonzero values, and then generating a Toeplitz matrix based on the corresponding discrete time signal. This method has the following advantages:

- The only operations involved are the computation of the DFT via the
Fast Fourier Transform (FFT) algorithm and the generation of a Toeplitz matrix, both of which are computationally efficient.

- The practical constraints on the covariance matrix, \( R \), are shown to be fulfilled in the general case.

- The beampattern matching performance is similar to that achieved by iterative methods.

Second, we present a novel radar architecture that sums phase-shifted orthogonal signals in order to match the desired beampattern. This architecture provides a number of advantages over currently existing methods of waveform design, which are:

- The architecture does not require the generation of partially correlated waveforms or the use of iterative methods to do so, allowing for rapid generation of arbitrary beampatterns.

- The waveforms can be designed directly, allowing the user to omit the usual first step of designing the covariance matrix, \( R \).

- The achieved transmit beampattern matches exactly the theoretical beampattern calculated by the first algorithm.

The remainder of this chapter is organized as follows. Section 2.2 describes the MIMO radar signal model and the problem setup. In Section 2.3, previous methods of solution are described. Section 2.4 describes the proposed method for covariance matrix design and Section 2.5 details the design of the actual transmit signals, which does not require the synthesis of the covariance matrix. Section 2.6 provides the theoretical computational complexity of the proposed algorithms. Simulation results are shown in Section 2.7, with conclusions given in Section 2.8.
2.2 Problem Formulation

In this section, the signal model for a MIMO radar with co-located antennas is developed. The power received at a target in the far field is defined, and the formulation for the transmit beampattern matching problem is given.

Consider a MIMO radar system with \( N_T \) co-located transmit antennas having interelement spacing \( d \) and transmission wavelength \( \lambda \). Define the transmitted baseband signal vector as

\[
\mathbf{x}(n) = \begin{bmatrix} x_1(n) & x_2(n) & \ldots & x_{N_T}(n) \end{bmatrix}^T.
\] (2.1)

Assuming the transmitted probing signals are narrowband and that the propagation is nondispersive, the signal received by a target located at an angle \( \theta \) at time \( n \) can be written as

\[
\mathbf{r}(n; \theta) = \mathbf{a}_T^H(\theta)\mathbf{x}(n),
\] (2.2)

where \((\cdot)^H\) denotes the conjugate transpose and \( \mathbf{a}_T(\theta) \) represents the transmit steering vector, given by

\[
\mathbf{a}_T(\theta) = \begin{bmatrix} 1 & e^{-j \frac{2\pi d}{\lambda} \sin(\theta)} & \ldots & e^{-j \frac{2\pi (N_T-1) d}{\lambda} \sin(\theta)} \end{bmatrix}^T.
\] (2.3)

The transmitted power at location \( \theta \) can then be found as

\[
P_t(\theta) = \mathbb{E} \{ \mathbf{a}_T^H(\theta)\mathbf{x}(n)\mathbf{x}^H(n)\mathbf{a}_T(\theta) \} = \mathbf{a}_T^H(\theta)\mathbf{R}\mathbf{a}_T(\theta),
\] (2.4)

where \( \mathbf{R} \) is the covariance matrix of the transmitted waveforms and \( \mathbb{E} \{ \cdot \} \) denotes the expectation operator. The objective is to design \( \mathbf{R} \) such that the transmitted power matches a desired beampattern as closely as possible while fulfilling the following
constraints

\[ C_1 : \mathbf{R} \geq 0 \]

\[ C_2 : \mathbf{R}(m, m) = c, \ m = 1, 2, \ldots, N_T, \]

where \( c \) is a constant equal to the power transmitted by each antenna. \( C_1 \) denotes the positive semidefinite constraint and \( C_2 \) denotes uniform elemental power constraint. Instead of requiring equal power transmission from each antenna, \( C_2 \) may be replaced by the total power constraint

\[ C_3 : \text{trace} (\mathbf{R}) = cN_T. \]

An intuitive problem setup, which is often employed in the literature, is that of matching some desired beampattern as closely as possible in a least-squares sense, yielding the cost function

\[
J(\mathbf{R}) = \frac{1}{L} \sum_{l=1}^{L} \left( \mathbf{a}_T^H(\theta_l)\mathbf{R}\mathbf{a}_T(\theta_l) - \alpha P_d(\theta_l) \right)^2,
\]

where \( P_d(\theta_l) \) is the desired beampattern defined over the grid points \( \{\theta_l\}_{l=1}^{L} \), and \( \alpha \) is a scaling factor. The desired beampattern may be obtained either from a priori knowledge about the scene, or using the generalized likelihood ratio test as described in [17]. The resulting optimization problem is

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{L} \sum_{l=1}^{L} \left( \mathbf{a}_T^H(\theta_l)\mathbf{R}\mathbf{a}_T(\theta_l) - \alpha P_d(\theta_l) \right)^2 \\
\text{subject to} & \quad \mathbf{R} \geq 0 \\
& \quad \mathbf{R}(m, m) = c, \ m = 1, 2, \ldots, N_T.
\end{align*}
\]
2.3 Previous Work

In this section, previous methods of solution are described for both the transmit covariance matrix design (Section 2.3.1) as well as transmit signal design (2.3.2).

2.3.1 Covariance Matrix Design

Previous work utilizes optimization techniques to minimize a variety of cost functions similar to (2.6). The work in [17] formulates this problem as a Semidefinite Quadratic Program (SQP) with a cost function involving a beampattern matching term and a cross-correlation term. The resulting cost function is

\[ J(R) = \frac{1}{L} \sum_{l=1}^{L} w_l (a_H^T(\theta_l)R a_T(\theta_l) - \alpha P_d(\theta_l))^2 + \frac{2w_c}{K^2 - K} \sum_{k=1}^{\tilde{K}} \sum_{p=k+1}^{\tilde{K}} |a_H^T(\theta_k)R a_T(\theta_p)|^2, \]

(2.7)

where \( w_l \) is the weighting at each angle and \( w_c \) is the weight for the cross-correlation term. Minimization of this cost function can be achieved using freely available software [26], where the constraints \( C_1 \) and \( C_2 \) can be trivially added.

In [20], the authors present unconstrained cost functions that can be solved using gradient-descent methods and do not necessarily rely on a quadratic cost function. In this case, \( C_1 \) is fulfilled by solving for the square-root matrix \( U \), such that \( R = U^H U \), and \( C_2 \) is fulfilled via a transformation to a hyperspherical coordinate system in order to force the norm of each column of \( U \) to be equal to \( c \). The spherical transformation is represented by the parameterization of \( U \) by \( \psi = [\psi_{21} \psi_{31} \psi_{32} \ldots \psi_{M,M-1}]^T \) (see [20] for further details). In this case, the cost function (2.7) is replaced by

\[ J_1(\Theta) = \frac{1}{L} \sum_{l=1}^{L} (P(\psi, \theta_l) - \alpha P_d(\theta_l))^2, \]

(2.8)

where \( \Theta = [\psi^T \alpha]^T \) and \( P(\psi, \theta_l) = a_H^T(\theta_l)U^H(\psi)U(\psi)a_T(\theta_l) \). The authors also
demonstrate the superior performance achieved by the 1-norm minimization by minimizing the cost function

\[ J_2(\Theta) = \frac{1}{L} \sum_{l=1}^{L} |P(\psi, \theta_l) - \alpha P_d(\theta_l)|. \quad (2.9) \]

While many existing solutions rely on iterative methods to design the covariance matrix, closed-form solutions exist as well. In [16], rather than minimizing a cost function, the authors directly provide the cross-correlation between signals, which can be used to generate the covariance matrix. The cross-correlation is given as

\[ r_{lm} = \frac{\sin[\pi \beta (l - m)]}{\sin[\pi \beta (l - m)/N_T]} \]. \quad (2.10) \]

In this case, the beamwidth of the resulting pattern is proportional to \( \beta \), which can be varied from 0 to 1. The other closed-form method for generating covariance matrices appears in [20], but serves only as an approximation to the desired beampattern. The method utilizes a summation of phased-array beams at the angles within the desired ROIs, yielding the covariance matrix

\[ \mathbf{R} = \frac{1}{L} \sum_{l=1}^{L} P_d(\theta_l) \mathbf{a}_T(\theta_l) \mathbf{a}_T^H(\theta_l). \quad (2.11) \]

While both of these methods provide closed-form solutions that fulfill both \( C_1 \) and \( C_2 \), neither results in performance that is comparable to that of the iterative methods. The motivation of the proposed work is therefore to provide a closed-form solution to the transmit beampattern matching problem that provides similar performance to iterative methods.
2.3.2 Transmit Signal Design

While the design of the transmit covariance matrix is straightforward and allows the use of convex optimization routines, the design of the actual transmit signals which realize the desired covariance matrix is less straightforward. In [22], the authors present a cyclic algorithm to draw signals from an infinite alphabet. Let the matrix of signals transmitted from each antenna be $X$. In this case, the unconstrained signal matrices that realize the desired $R$ fulfill

$$\frac{1}{\sqrt{S}}X^* = R^{1/2}U^*,$$  \hspace{1cm} (2.12)

where $S$ is the number of transmitted symbols. Let the set $C$ denote the set of signal matrices that fulfill some constraints (such as having unity PAPR). Waveform design is achieved through the minimization of the optimization problem

$$\min_{X \in C; U} \|X - \sqrt{S}UR^{1/2}\|^2.$$  \hspace{1cm} (2.13)

The given minimization is performed via a cyclic algorithm that alternates between minimizing over $X$ and $U$. Details are given in the reference [22].

The other method of waveform design is that of [23] and [24], where the authors present a method of mapping Gaussian random variables onto BPSK and QPSK signals. In both cases, the authors find a relationship between the cross correlations of Gaussian and BPSK random variables. Once the Gaussian signal vectors are achieved, BPSK transmit symbols can be obtained by simply taking the sign of the designed Gaussian signals. QPSK symbols can be obtained via a similar mapping. In order to match the desired covariance matrix, particle swarm optimization is employed.
2.4 Fourier-Based Covariance Matrix Design

In this section, we describe the proposed method of covariance matrix design, which exploits the DFT. By specifying a rectangular window (or set of windows) of varying width in the frequency domain, the user can equivalently achieve beampatterns of varying widths in the spatial domain.

To establish the relationship between the rectangular window in the frequency domain and the desired beampattern, let \( \{H(k)\}_{k=0}^{N_T-1} \) denote the frequency domain samples and \( \{h(n)\}_{n=0}^{N_T-1} \) be the corresponding time domain samples obtained after applying the inverse DFT. The relationship between them can be described by the \( N_T \)-point DFT operation as

\[
H(k) = \sum_{n=0}^{N_T-1} h(n) e^{-j2\pi kn/N_T} \tag{2.14}
\]

\[
h(n) = \frac{1}{N_T} \sum_{k=0}^{N_T-1} H(k) e^{j2\pi kn/N_T}. \tag{2.15}
\]

Using (2.14) and (2.15), the following lemma can be obtained.

**Lemma 1.** If \( H(k) \) is real and nonnegative and \( \mathbf{R} \) is the Toeplitz matrix formed using the samples \( \{h(n)\} \) as

\[
\mathbf{R} = \begin{bmatrix}
    h(0) & h(1) & \cdots & h(N_T - 1) \\
    h^*(1) & h(0) & \cdots & h(N_T - 2) \\
    h^*(2) & h^*(1) & \cdots & h(N_T - 3) \\
    \vdots & \vdots & \ddots & \vdots \\
    h^*(N_T - 1) & h^*(N_T - 2) & \cdots & h(0)
\end{bmatrix},
\]

then it can be easily proved that \( \mathbf{R} \) will be positive semidefinite with maximum value along the diagonal.

**Proof.** We begin by transforming \( \mathbf{R} \) into a more convenient notation. Stack the
elements \( \{h(n)\} \) to obtain the column vector \( \mathbf{h} \) with \( n \)th element

\[
[h]_n = \frac{1}{N_T} \sum_{k=0}^{N_T-1} H(k) e^{j2\pi kn/N_T}.
\]  \hspace{1cm} (2.16)

It can be noted here that for any window choice \([\mathbf{h}]_0\) is the maximum value over all \( n \). Thus the resulting matrix has maximum value along the main diagonal. Let \( \mathbf{h}_k \) be the contribution to the summation in (2.16) due to the \( k \)th value, i.e.,

\[
[h_k]_n = \frac{1}{N_T} H(k) e^{j2\pi kn/N_T}.
\]  \hspace{1cm} (2.17)

Then

\[
\mathbf{h} = \sum_{k=0}^{N_T-1} \mathbf{h}_k,
\]  \hspace{1cm} (2.18)

and it is shown in Appendix 2.9.1 that

\[
\mathbf{h}_k^H\mathbf{h}_l = \frac{1}{N_T} H(k)^2 \delta_{kl},
\]  \hspace{1cm} (2.19)

where \( \delta_{kl} \) is the Kronecker delta. Next, let \( \{p_i\}_{i=1}^P \) be the set of values of \( k \) for which \( H(k) \) is nonzero. In this case, \( \mathbf{h} \) can be rewritten as

\[
\mathbf{h} = \sum_{i=1}^P \mathbf{h}_{p_i},
\]  \hspace{1cm} (2.20)

and it can be shown by inspection that the resulting Toeplitz matrix created using \( \mathbf{h} \) is equivalent to

\[
\mathbf{R} = N_T \sum_{i=1}^P \mathbf{h}_{p_i}^H \mathbf{h}_{p_i}^T.
\]  \hspace{1cm} (2.21)

For any arbitrary vector \( \mathbf{g} \) of corresponding length, we can write

\[
\mathbf{g}^H \mathbf{R} \mathbf{g} = N_T \sum_{i=1}^P |\mathbf{g}^H \mathbf{h}_{p_i}|^2,
\]  \hspace{1cm} (2.22)
which is real and nonnegative. Thus the proposed matrix is positive semidefinite for all real and nonnegative $H(k)$. 

From Lemma 1 it can be seen that the proposed matrix fulfills the necessary constraints. The remaining question is how to best design $\mathbf{R}$ to achieve the desired beampattern. We now introduce a second lemma and describe the relationship between the frequency-domain window and the transmitted spatial power.

**Lemma 2.** If $H(k)$ is real and $\mathbf{R}$ is the Toeplitz matrix formed using the samples $\{h(n)\}$ as in Lemma 1, then it can be easily proved that

$$
e^{H(k)} \mathbf{Re}(k) = N_T H(k), \tag{2.23}$$

where

$$
e(k) = \begin{bmatrix} 1 & e^{-j\frac{2\pi k}{N_T}} & \ldots & e^{-j\frac{2\pi k(N_T-1)}{N_T}} \end{bmatrix}^T \tag{2.24}$$

is the Fourier vector corresponding to frequency $k$.

Please see Appendix 2.9.2 for proof.

Using Lemma 1, we can write

$$
\sum_{k=0}^{N_T-1} \left( e^{H(k)} \mathbf{Re}(k) - \frac{N_T}{\bar{h}(0)} H(k) \right)^2 = 0, \tag{2.25}
$$

where the matrix $\mathbf{\bar{R}} = \frac{\mathbf{R}}{\bar{h}(0)}$ is the normalized covariance matrix. Since $e(k)$ is similar to the steering vector $\mathbf{a}_T(\theta_k)$ and $\mathbf{\bar{R}}$ is a covariance matrix, the expression (2.25) is similar to the cost function (2.7) of the beampattern design problem to synthesize the covariance matrix. Therefore, considering $H(k)$ as the desired transmit beampattern and $e^{H(k)} \mathbf{Re}(k)$ as the designed beampattern at discrete point $k$, the transmit beampattern design problem can be mapped onto (2.25) to find the waveform covariance matrix.
In order to map the discrete frequency point, \( k \), onto the positive and negative spatial locations of \( \theta_k \), note that \( k \) denotes both positive and negative frequency points. The following relationship can be used to map the frequency components onto the spatial domain for an even number of transmit antennas

\[
\theta_k = \begin{cases} 
\sin^{-1}\left(\frac{\lambda k}{dN_T}\right), & k = 0, \ldots, \frac{N_T}{2} \\
-sin^{-1}\left(\frac{\lambda(N_T-k)}{dN_T}\right), & k = \frac{N_T}{2} + 1, \ldots, N_T - 1
\end{cases}.
\]  

(2.26)

Similarly, the mapping for an odd number of transmit antennas is

\[
\theta_k = \begin{cases} 
\sin^{-1}\left(\frac{\lambda k}{dN_T}\right), & k = 0, \ldots, \lfloor \frac{N_T}{2} \rfloor \\
-sin^{-1}\left(\frac{\lambda(N_T-k)}{dN_T}\right), & k = \lceil \frac{N_T}{2} \rceil + 1, \ldots, N_T - 1
\end{cases}.
\]  

(2.27)

where \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \) denote the floor and ceiling, respectively.

Given the above mappings, it can be seen that a window in the frequency (\( k \)) domain results in a window of proportional width in the spatial (\( \theta \)) domain. In the two extremes of a single nonzero point at \( k = 0 \) and a vector of all non-zero points, the beampattern results in the phased-array and omnidirectional patterns, respectively. Fig. 2.1 shows the frequency domain and spatial domain power in the case of \( N_T = 10 \) with half-wavelength interelement spacing and a window with \( H(k) = 1, k = 0, 1, 9 \). The resulting beampattern has amplitude 10 at 0° and ±11.54°.

While the beampattern matches (2.23) for spatial values corresponding to integer values of \( k \), the function does not describe the beampattern for other values of \( \theta \). We now demonstrate that choosing the positive coefficients in the window function is equivalent to adding phased-array beams with centers at the given locations. We begin by noting that a covariance matrix to generate \( P \) phased-array beams with centers located at \( \{\theta_i\}_{i=1}^P \), can be equivalently designed by creating a Toeplitz matrix
with the first row

$$\mathbf{a} = \sum_{i=1}^{P} \mathbf{a}_T^*(\theta_i)$$

$$= \sum_{i=1}^{P} \left[ 1 \ e^{j \frac{2 \pi d}{\lambda} \sin(\theta_i)} \ ... \ e^{j \frac{2(N_T-1)d}{\lambda} \sin(\theta_i)} \right]^T.$$  \hspace{1cm} (2.28)

or through a summation of phased-array covariance matrices

$$\mathbf{R} = \sum_{i=1}^{P} \mathbf{a}_T(\theta_i) \mathbf{a}_T^H(\theta_i).$$  \hspace{1cm} (2.29)

It has been shown [25] that for receive beamformers, many widely used methods are composed of a summation of conventional (uniformly weighted) beamformers with centers at various locations. On the transmit side, the conventional beamformer corresponds to the phased-array beam (see Appendix 2.9.3). The method presented in [20] (defined by (2.11)) shows that the sum of phased-array beams provides a suitable guess at the covariance matrix needed to match a given beampattern. However, in this case the sidelobes levels are high and the main lobe suffers from slow roll-off due
to the high number of beams in the summation. Thus, performance benefits can be achieved by choosing a lower number of beams to sum. For the frequency-domain window with \( P \) unity coefficients at locations \( \{p_i\}_{i=1}^P \), the transformed coefficients are

\[
h(n) = \frac{1}{N_T} \sum_{i=1}^P e^{j2\pi p_i n/N_T},
\]

\[
= \frac{1}{N_T} \sum_{i=1}^P e^{j2\pi d \lambda \sin(\theta_i)}, \quad (2.30)
\]

where \( \{\theta_i\}_{i=1}^P \) is the set of locations corresponding to \( \{p_i\}_{i=1}^P \) and is found by solving \( (2.26) \). Vectorizing over all values of \( n \), we obtain the frequency and spatial domain vectors

\[
h = \frac{1}{N_T} \sum_{i=1}^P \begin{bmatrix} e^{j2\pi p_i n/N_T} & \cdots & e^{j2\pi p_i (N_T - 1) n/N_T} \end{bmatrix}^T
\]

\[
= \frac{1}{N_T} \sum_{i=1}^P \begin{bmatrix} e^{j2\pi \frac{d \lambda}{\lambda} \sin(\theta_i)} & \cdots & e^{j2\pi \frac{d \lambda (N_T - 1)}{\lambda} \sin(\theta_i)} \end{bmatrix}^T. \quad (2.31)
\]

It is easily seen that \( h \) as described in \( (2.31) \) corresponds to a scaled version of \( (2.28) \). For this reason, we conclude that the method of choosing beam locations using \( P \) DFT coefficients is equivalent to a sum of \( P \) phased array beams. Since the DFT coefficients represent mutually orthogonal frequencies, the resulting beams are placed at the nulls of the other beams in the sum, resulting in a smooth function within the window and overlapping sidelobes and nulls outside the ROI. In order to choose the values of \( k \) that define \( \{p_i\}_{i=1}^P \) for a given ROI, Algorithm 1 can be employed.

The main drawback of the method as proposed is that there are only \( N_T \) degrees of freedom available for beampattern design. Therefore, arrays with a low number of transmit antennas will only be able to transmit beams with a limited number of widths. However, with recent advances in sensor technology, the number of transmit antennas in systems has grown significantly. In such systems, the achievable resolution
Algorithm 1 Method of choosing \( \{p_i\}_{i=1}^{P} \) for a given ROI

\[
\begin{align*}
\text{input: } & P_d(\theta) \\
\theta_{\text{max}} & = \max\{\theta \in \text{ROI}\} \\
\theta_{\text{min}} & = \min\{\theta \in \text{ROI}\} \\
\text{solve } (2.26) \text{ or } (2.27) \text{ to obtain } k_{\text{max}} \text{ corresponding to } \theta_{\text{max}} \text{ and } k_{\text{min}} \text{ corresponding to } \theta_{\text{min}} \\
k^+ & \leftarrow \lfloor k_{\text{max}} \rfloor \\
k^- & \leftarrow \lceil k_{\text{min}} \rceil \\
\{p_i\} & = \{k \in \mathbb{Z} \mid k^- \leq k \leq k^+\}
\end{align*}
\]

will be sufficient. The available resolution can be characterized by the null-to-null beamwidth, \( BW_{NN} \). For a single phased-array beam, it is proved in Appendix 2.9.3 that

\[
BW_{NN-P_A} = 2\sin^{-1}\left(\frac{2\lambda}{dN_T}\right).
\]

This describes the minimum beamwidth achievable for any radar setup. Let \( \theta_{k^+} \) be the greatest value of \( \theta \) in the ROI, corresponding to \( k = k^+ \) as defined by (2.26) or (2.27). Similarly, let \( \theta_{k^-} \) be the smallest value of \( \theta \) in the ROI, corresponding to \( k = k^- \). Since the proposed method involves a summation of phased-array beams, the overall beamwidth is easily found to be

\[
BW_{NN} = \theta_{k=k^+ + 1} + \theta_{k=k^- - 1}.
\]

For example, for the beampattern shown in Fig. 2.1, \( k^+ = 1 \) and \( k^- = 9 \), and therefore

\[
BW_{NN} = \theta_{k=2} + \theta_{k=8} \approx 47.16^\circ.
\]

The other problem with the proposed method is that the resulting covariance matrix is rank-deficient and equal to the number of nonzero values of \( H(k) \) (see proof of Lemma 1), reducing the total number of resolvable targets. However, since
beamforming is often performed to account for some ambiguity in the location of a single target or limited number of targets [21], we conclude that the rank deficiency is a worthwhile tradeoff given the computational gains achieved by the proposed method.

2.5 Fourier-Based Transmit Waveform Design

In this section, a method of directly designing transmit waveforms to achieve the beampattern which is obtained by synthesizing $\mathbf{R}$ in the previous section is described. While it is possible to stop with the covariance matrix designed in Section 2.4 and use the techniques presented in [22, 23, 24] to select waveforms, the architecture presented here does not rely on iterative methods of solution and can thus be employed for rapid transmission of arbitrary beampatterns.

Consider an arbitrary $H(k)$ used to define a desired transmit window and its corresponding vector $\mathbf{h}$ as described by (2.31). As demonstrated in (2.21) and (2.19),
the transmit covariance matrix can be written as a sum of $P$ rank-1 matrices defined by mutually orthogonal column vectors. Because of this fact, the beampattern can be achieved by transmitting a combination of $P$ orthogonal sets of symbols drawn from any modulation scheme (e.g., BPSK or QPSK). Let $x_i(n)$ represent the symbol at time $n$ that is weighted by the vector $h_{pi}$, and let $h_{pi}(m)$ denote the $m$th element of $h_{pi}$. The proposed architecture, as shown in Fig. 2.2, transmits the following signal from antenna $m$ at time $n$

$$v_m(n) = \sum_{i=1}^{P} x_i(n)h_{pi}(m). \quad (2.35)$$

The average power transmitted from antenna $m$ is then found to be

$$E\{v_m(n)v_m^*(n)\} = E \left\{ \left| \sum_{i=1}^{P} x_i(n)h_{pi}(m) \right|^2 \right\}$$

$$= E \left\{ \sum_{i=1}^{P} \sum_{l=1}^{P} x_i(n)x_l^*(n)h_{pi}(m)h_{pl}^*(m) \right\}$$

$$= \sum_{i=1}^{P} \sum_{l=1}^{P} E\{x_i(n)x_l^*(n)\} h_{pi}(m)h_{pl}^*(m)$$

$$= \sum_{i=1}^{P} h_{pi}(m)h_{pi}^*(m)$$

$$= \frac{P}{N_1^2}, \quad (2.36)$$

where we have made use of (2.19) and assumed the symbols to be orthonormal. From (2.36), one can see that the transmitted power is independent of the antenna number, demonstrating that the uniform elemental power constraint is fulfilled. The instantaneous power transmitted by antenna $m$ at time $n$ is

$$P_{inst}(m; n) = v_m(n)v_m^*(n) = \sum_{i=1}^{P} \sum_{l=1}^{P} x_i(n)x_l^*(n)h_{pi}(m)h_{pl}^*(m). \quad (2.37)$$
When transmitting BPSK symbols, the peak instantaneous power occurs when all symbols are equal to one, resulting in

\[
P_{\text{peak}}(m) = \sum_{i=1}^{P} \sum_{l=1}^{P} h_{p_i}(m)h_{p_l}^*(m) = \frac{1}{N_T^2} \sum_{i=1}^{P} \sum_{l=1}^{P} e^{j2\pi m(p_i-p_l)/N_T}.
\] (2.38)

From (2.38), it can be seen that the maximum instantaneous power occurs when \(m = N_T\), in which case the instantaneous power is \(P^2/N_T^2\) and the resulting PAPR is \(P\). However, for \(m \neq N_T\), the PAPR is less than \(P\).

Using this setup, the signal received by a target located at angle \(\theta\) at time \(n\) becomes

\[
r(n; \theta) = \sum_{i=1}^{P} h_{p_i}^T a_T(\theta)x_i(n).
\] (2.39)

Let \(S\) be the total number of symbols transmitted and \(x_i\) be the column vector of \(S\) symbols. Define the symbol and weight matrices as

\[
X = \begin{bmatrix} x_1 & x_2 & \ldots & x_P \end{bmatrix}^T_{P \times S}
\] (2.40)

and

\[
H = \begin{bmatrix} h_{p_1} & h_{p_2} & \ldots & h_{p_P} \end{bmatrix}^T_{P \times N_T},
\] (2.41)

respectively. Vectorizing the received symbols, the received signal from angle \(\theta\) can be written as

\[
r(\theta) = \begin{bmatrix} r(0; \theta) & r(1; \theta) & \cdots & r(S-1; \theta) \end{bmatrix}^T = X^T H a_T(\theta).
\] (2.42)
The power delivered to angle $\theta$ is then

$$P_t(\theta) = E \{ a_T^H(\theta) H^H X^* X^T H a_T(\theta) \}$$

$$= a_T^H(\theta) H^H H a_T(\theta). \tag{2.43}$$

The resulting system is a cross between phased-array and MIMO systems in which the number of uncorrelated waveforms is $P \leq N_T$. The proposed system provides benefits over the waveform design methods presented in [22, 23, 24] in that it does not require the generation of partially correlated symbols and therefore has a much lower computational cost. Rather, the transmitted signals can be designed directly, and the initial step of designing the covariance matrix can be omitted. In addition, transmission of truly orthogonal signals (e.g., BPSK symbols drawn from Hadamard code sequences) allows the transmitted beampattern to match the theoretical beampattern achieved by the covariance design procedure in the previous section.

### 2.6 Computational Complexity

#### 2.6.1 Covariance Matrix Design

The only operations required for the proposed covariance matrix design are the FFT and the generation of a Toeplitz matrix. The complexity of the $N$-point FFT is well known and equal to $O(N \log(N))$ computations. The complexity of generating a Toeplitz matrix is considered negligible, yielding an overall complexity of $O(N_T \log(N_T))$. As a comparison, the SQP algorithm from [17] has a complexity of $O \left( \log \left( \frac{1}{\eta} \right) N_T^{3.5} \right)$ for a prefixed accuracy of $\eta$ [23].
2.6.2 Transmit Signal Design

Once the orthogonal symbols are obtained, the proposed radar architecture requires $P$ complex multiplications and $P$ complex additions per symbol for each antenna. Therefore, the total number of operations required for $S$ symbols transmitted from $N_T$ antennas is $SPN_T$ real multiplications and $SPN_T$ real additions. Since this method does not require $R$ to be generated before the transmit signals are designed, this represents the total computational complexity of the architecture. In comparison, the algorithm proposed in [22] requires $O(S + 3SN_T^2)$ operations per iteration after generating $R$ (incurring the computational cost described above). The efficient algorithm presented in [23] requires an overall complexity of \( \sum_{m=3}^{N_T} m + \frac{N_T(N_T^2 - N_T)}{2} + L \left( \frac{(N_T-1)(N_T^2 - N_T)}{2} + (2N_T^2 + 4N_T) \right) \) operations per iteration and $O(N_T^2 + SN_T^2)$ real multiplications to find both the covariance matrix and the BPSK symbols, where $L$ is the number of elements in the grid used to define $P_d(\theta)$.

2.7 Simulation Results

In this section we present numerical examples to demonstrate the performance of the design methods described. Simulations assume a uniform linear array with half-wavelength interelement spacing and a mesh grid with spacing of 0.1°.

We begin by demonstrating the performance of the method for covariance matrix design presented in Section 2.4. Fig. 2.3 shows the resulting beampattern with $N_T = 10$ transmit antennas for the ROI defined by $\theta \in [-30°, 30°]$. Employing Algorithm 1, we obtain \( \{p_i\} = \{0, 1, 2, 8, 9\} \) and set $H(k) = 1$ for these values of $k$. The solution found using the SQP as presented in [17] is also included for comparison. The total transmit power is normalized to unity. The theoretical $BW_{NN}$ for the proposed method is

\[
BW_{NN} = \theta_{k=3} + \theta_{k=7} \approx 73.7°, \tag{2.44}
\]
Comparison of beampatterns achieved by proposed method of covariance matrix design with $H(k) = 1$, $k = 0, 1, 2, 8, 9$ and SQP method. The ROI is $\theta \in [-30^\circ, 30^\circ]$ and $N_T = 10$.

which matches the simulated value exactly. For the SQP method, the resulting $BW_{NN} \approx 74.6^\circ$. It can be seen that the proposed method provides similar performance to the iterative solution. Define the Mean Square Error (MSE) as

$$MSE = \frac{1}{L} \sum_{l=1}^{L} |P(\theta_l) - \alpha \phi(\theta_l)|^2,$$

where $\alpha$ is found using the SQP method. The resulting MSEs are 0.0322 for the proposed method and 0.0311 for the SQP method. Thus, we conclude that the proposed method provides performance which is comparable to that achieved by iterative methods while incurring a fraction of the computational cost. Fig. 2.4 shows the resulting beampattern for $N_T = 50$ with ROIs defined by $\theta \in [-44^\circ, -15^\circ] \cup [15^\circ, 44^\circ]$. Employing Algorithm 1 results in $H(k) = 1$, $k = 8 - 18, 34 - 44$. The resulting MSEs are 0.0230 and 0.0175 for the proposed and SQP methods, respectively. Fig. 2.5 shows the computational complexity (as defined in Section 2.6) as a function of the number of transmit antennas for both the proposed and SQP methods with $\eta = 0.0311$. The results are shown in subfigures due to the difference in scale, and
Figure 2.4: Beampattern achieved through covariance matrix design for $N_T = 50$ with $H(k) = 1$, $k = 8 - 18, 34 - 44$. The corresponding ROIs are $\theta \in [-44, -15] \cup [15, 44]$.

The figures demonstrate the clear computational advantage of the proposed method, especially as the number of transmit antennas becomes large.

Next, we demonstrate the performance achieved by the proposed radar architecture in Section 2.5. Fig. 2.6 shows the beampattern generated using the proposed method with orthogonal BPSK symbols drawn from Hadamard code sequences of length 128. The resulting beampattern formed using the CE waveforms described in [22] with PAPR = 1 is included for comparison (note that these waveforms come from an infinite alphabet). The desired beampattern is the same as in Fig. 2.3, and the resulting MSEs obtained by averaging over 100 Monte Carlo trials are 0.0322 for the proposed method and 0.0328 for the SQP. Thus, the proposed method provides a lower MSE and utilizes symbols from a finite alphabet while incurring a much lower computational cost. Fig. 2.7 shows the MSE as a function of the number of samples when transmitting via the proposed method and the method presented in [22]. Since it is possible to generate truly orthogonal BPSK symbols, the proposed method results in a uniform MSE regardless of the number of samples. The proposed method also requires a negligible computational time compared to the iterative methods used
in [22, 23, 24]. Moreover, the proposed architecture does not require the user to generate the theoretical covariance matrix prior to designing the actual waveforms, which can present a large computational cost as demonstrated in Fig. 2.5.

We conclude this section by demonstrating the receive beampattern after applying the adaptive MVDR or Capon beamformer. Consider the scenario in which there are three targets of interest located at 0° and ±15° and one interfering target located at −50° that we desire to suppress. Probing signals are transmitted using either the proposed radar architecture or method presented in [22] with the same desired beampattern as in Figs. 2.2 and 2.6. The received signal is corrupted by white Gaussian noise with a variance equal to 0.01. Fig. 2.8 shows the average resulting beampatterns from the two methods after transmitting $S = 16$ symbols over 1000 Monte Carlo simulations. Due to the reliance on perfectly uncorrelated symbols, the proposed method results in lower sidelobe levels as well as greater attenuation of the interfering target (about 3 dB in this case). Note that the CE waveforms also incur a downward bias of 6.3 dB for the target located at −15° and 4.93 dB for the target located at 15°. In contrast, the proposed architecture results in biases of 1.58 dB and
Figure 2.6: Beampatterns achieved by transmitting symbols via the proposed architecture and the CE waveforms obtained using the cyclic algorithm from [22].

1.15 dB. Although not pictured, similar results are obtained for $S = 256$ transmitted symbols.

2.8 Conclusion

We have demonstrated a closed-form method of covariance matrix design for the MIMO transmit beamforming problem that exploits the properties of DFT coefficients. The resulting covariance matrix fulfills both the positive semidefinite and uniform elemental power constraints, is computationally efficient, and results in similar performance to that achieved by iterative solutions. We have also demonstrated a radar architecture that can be used with a set of orthogonal signals to match the desired beampattern. The resulting beampattern matches the theoretical pattern exactly, resulting in superior MSE performance compared to existing methods. The architecture itself is computationally efficient and allows the user to forego the usual preliminary step of designing the covariance matrix.
Figure 2.7: MSE of transmitted power as a function of the number of samples when transmitting symbols via the proposed architecture and the cyclic algorithm from [22] with PAPR = 1.

2.9 Appendices

2.9.1 Orthogonality of $h_k$’s

Consider $h_k$ as defined by (2.17). This yields the resulting inner product

$$h_k^H h_l = \frac{H(k)H(l)}{N_T^2} \sum_{n=0}^{N_T-1} e^{j2\pi(l-k)n/N_T}$$

$$= \frac{H(k)H(l)}{N_T^2} \frac{1 - e^{j2\pi(l-k)}}{1 - e^{j2\pi(l-k)/N_T}}$$

$$= \frac{H(k)H(l) \sin(\pi(l-k))}{N_T^2 \sin\left(\frac{\pi}{N_T}(l-k)\right)}. \quad (2.46)$$
Figure 2.8: Receive beampatterns after applying the MVDR beamformer for multiple targets of interest using the proposed architecture and the CE waveforms generated by [22].

The above result is clearly zero for $l \neq k$. In the case of $l = k$, we invoke L'Hopital’s rule to find the limit

$$
\lim_{(l-k) \to 0} \frac{H(k)H(l)}{N_T^2} \frac{\sin(\pi(l-k))}{\sin\left(\frac{\pi}{N_T}(l-k)\right)} = \lim_{(l-k) \to 0} \frac{H(k)H(l)}{N_T^2} \frac{\pi \cos(\pi(l-k))}{\pi \cos\left(\frac{\pi}{N_T}(l-k)\right)} = \frac{H(k)^2}{N_T}.
$$

The final result is then

$$
h_k^H h_l = \frac{1}{N_T} H(k)^2 \delta_{kl},
$$

where $\delta_{kl}$ denotes the Kronecker delta.
2.9.2 Proof of Lemma 2

Consider the k-space steering vector defined in (2.24). Using the fact that $h(n) = h^*(N_T - n)$, it follows that the vector $e^H(k)R$ has elements

$$\left[ e^H(k)R \right]_i = \sum_{n=0}^{N_T-1} h^*(n)e^{j\frac{2\pi k(n+i)}{N_T}}, \; i = 0, \ldots, N_T - 1. \quad (2.48)$$

Thus, the power in k-space is

$$e^H(k)Re(k) = N_T \left( \sum_{n=0}^{N_T-1} h(n)e^{-j\frac{2\pi kn}{N_T}} \right)^*$$

$$= N_T H^*(k)$$

$$= N_T H(k) \quad (2.49)$$

2.9.3 $BW_{NN}$ for Phased-Array Beam

For the phased-array beam, the covariance matrix $R = 11^H$, where 1 represents the column vector of all ones. The resulting power is

$$P_t(\theta) = a_T^H(\theta)11^H a_T(\theta)$$

$$= \sum_{n=0}^{N_T-1} e^{j\frac{2\pi d\sin(\theta)}{\lambda}} \sum_{m=0}^{N_T-1} e^{-j\frac{2\pi d\sin(\theta)}{\lambda}}$$

$$= \left| \frac{1 - e^{j\frac{2\pi dN_T\sin(\theta)}{\lambda}}}{1 - e^{j\frac{2\pi d\sin(\theta)}{\lambda}}} \right|^2$$

$$= \left| \frac{\sin \left[ \frac{\pi d N_T \sin(\theta)}{\lambda} \right]}{\sin \left[ \frac{\pi d \sin(\theta)}{\lambda} \right]} \right|^2, \quad (2.50)$$

which is the familiar beampattern of the uniformly weighted ULA [25]. The nulls occur where the numerator is minimized and the denominator is not equal to zero. This is true when

$$\frac{\pi d}{\lambda} N_T \sin(\theta) = m\pi, \; m = 1, \ldots, \frac{N_T}{2}. \quad (2.51)$$
resulting in nulls at the locations

\[ \theta = \pm \sin^{-1} \left( \frac{\lambda m}{dN_T} \right), \quad m = 1, \ldots, \frac{N_T}{2}. \tag{2.52} \]
Chapter 3

Receive Beampattern Matching for MIMO Radar

3.1 Introduction

Receive beampattern design, or beamforming, has been a topic of interest in array processing for decades (see [25, 27] and the references therein). In the case of receive beamforming, the user wishes to receive signal power only from a desired region or set of regions of interest. Regions of interest could be chosen using a-priori knowledge about the surroundings or targets of interest. More recently, passive radar applications have arisen [28, 29, 30], in which the user does not have any control on the transmitter side; therefore the transmit beampattern cannot be designed. In such scenarios, the interference-plus-noise term can be significant, and radar performance is heavily dependent on the ability to suppress direct-path and multipath interference [28]. This type of suppression can be achieved through the use of receive beamforming.

In MIMO radar, the receive signal can be match-filtered at each receive antenna to obtain a virtual steering vector of increased length [12, 8], which can provide improved resolution and better beampattern matching performance. In this chapter, six methods for receive beampattern matching are proposed. We begin by formulating
a nonlinear least-squares problem, and then demonstrate two methods of solution to this optimization problem. We show that partitioning the received signal vector into subvectors and then multiplying each subvector with its corresponding weight vector results in improved performance. We then present an intuitive closed-form method, known as the Simple method, which serves as a good approximation to the solution. Finally, we show how the Simple method can be combined with the other methods, as well as itself, in order to provide reduced sidelobe levels.

While the methods presented are novel, it will be shown in simulations that they do not present any performance gains over the classic methods developed in the past. Moreover, nearly all of the proposed methods rely on iterative methods of optimization, resulting in a higher computational complexity than existing methods.

The remainder of this chapter is organized as follows. Section 3.2 describes a general MIMO system and the problem formulation for receive beampattern matching. Previous work in receive beampattern design is described in 3.3. Section 3.4 describes the proposed beampattern matching methods. Numerical examples are given in Section 3.5, including beampattern matching performance, SINR improvement, and a detection example. Some conclusions are given in Section 3.6.

### 3.2 Problem Formulation

Consider a MIMO radar system in which there are $N_T$ transmit antennas and $N_R$ receive antennas, with interelement spacing $d$ and transmission wavelength $\lambda$. The received signal at antenna $p$ and time index $n$ due to a target located at angle $\theta$ with complex reflection coefficient $\beta(\theta)$ can be written as

$$y_p(n) = \beta(\theta)e^{j\frac{2(p-1)\pi d}{\lambda}\sin(\theta)}\sum_{q=1}^{N_t} x_q(n)e^{j\frac{2(q-1)\pi d}{\lambda}\sin(\theta)} + \epsilon_p(n),$$

(3.1)
where $x_q(n)$ represents the baseband signal transmitted from antenna $q$ and $\epsilon_p(n)$ is circularly symmetric white Gaussian noise. Assuming the transmitted probing signals are narrowband and that the propagation is nondispersive, the received baseband data vector due to a target at location $\theta$ can be described by [31]

$$y(n) = \beta(\theta) a_R(\theta) a_T^T(\theta) x(n) + \epsilon(n), \quad (3.2)$$

where

$$y(n) = \begin{bmatrix} y_1(n) & y_2(n) & \ldots & y_{N_R}(n) \end{bmatrix}^T, \quad (3.3)$$

$$x(n) = \begin{bmatrix} x_1(n) & x_2(n) & \ldots & x_{N_T}(n) \end{bmatrix}^T, \quad (3.4)$$

$$\epsilon(n) = \begin{bmatrix} \epsilon_1(n) & \epsilon_2(n) & \ldots & \epsilon_{N_R}(n) \end{bmatrix}^T, \quad (3.5)$$

while

$$a_T(\theta) = \begin{bmatrix} 1 & e^{j \frac{2\pi d}{\lambda} \sin(\theta)} & \ldots & e^{j \frac{2(N_T-1)\pi d}{\lambda} \sin(\theta)} \end{bmatrix}^T \quad (3.6)$$

and

$$a_R(\theta) = \begin{bmatrix} 1 & e^{j \frac{2\pi d}{\lambda} \sin(\theta)} & \ldots & e^{j \frac{2(N_R-1)\pi d}{\lambda} \sin(\theta)} \end{bmatrix}^T \quad (3.7)$$

represent the transmit and receive steering vectors, and $(\cdot)^T$ denotes the transpose.

Samples received after matched-filtering at each antenna are correlated with the transmitted symbols. After correlation, by cascading all of the samples in one column, the $N_T N_R \times 1$ received signal vector can be written as

$$\tilde{y}(\theta) = \beta(\theta) a_T(\theta) \otimes a_R(\theta) + \tilde{\epsilon}, \quad (3.8)$$
where $\otimes$ denotes the Kronecker product and \( a_T(\theta) \otimes a_R(\theta) \) is the virtual steering vector corresponding to the target at location $\theta$ [31]. If

\[
v(\theta) = \beta(\theta) a_T(\theta) \otimes a_R(\theta), \tag{3.9}
\]

then the received power from the direction $\theta$, $P_r(\theta)$, is defined as

\[
P(\theta) = v^H(\theta)v(\theta), \tag{3.10}
\]

where $(\cdot)^H$ denotes the conjugate transpose.

In applications such as passive radar and medical imaging, it is desirable to receive power only from a given direction or set of directions defined by a beampattern, $P_d(\theta)$. To design such a beampattern, the proposed methods partition the virtual data vector, $v(\theta)$, into $K$ subvectors, where $K$ can range from 1 to $N_TN_R$. The subvectors can partially overlap each other. For maximum overlap, each subvector contains $M = N_TN_R - K + 1$ elements (e.g., the first subvector consists of elements 1 through $M$, the second consists of elements 2 through $M + 1$, etc.). This partitioning allows us to gain performance benefits such as reduced sidelobe levels and reduced data vector length. Each subvector is then multiplied with the corresponding weight vector to form a new vector of length $K$. Let the $i$th partition vector be denoted as $v_i(\theta)$ and the corresponding weight vector as $w_i$. Then the new modeled received signal vector of length $K$ corresponding to the target at location $\theta$ can be written as

\[
\bar{v}(\theta) = \begin{bmatrix} w_1^Hv_1(\theta) & w_2^Hv_2(\theta) & \cdots & w_K^Hv_K(\theta) \end{bmatrix}^T. \tag{3.11}
\]

The received power from direction $\theta$ becomes

\[
P_r(\theta) = \sum_{k=1}^{K} w_k^Hv_k(\theta)v_k^H(\theta)w_k. \tag{3.12}
\]
We define the desired beampattern, $P_d(\theta)$, over a mesh grid of $L$ locations. We then seek some set of weight vectors, $\{w_k\}$, $k = 1, ..., K$, such that the received power matches the desired beampattern as closely as possible in a least-squares sense

$$J = \min_{w_1, w_2, ..., w_K} \frac{1}{L} \sum_{l=1}^{L} |P_r(\theta_l) - P_d(\theta_l)|^2.$$  \hspace{1cm} (3.13)

Since the number of optimization variables is equal to $KM$, partitioning increases the computational complexity of obtaining a solution. However, $KM$ is maximized for $K = \frac{M+1}{2}$, resulting in a maximum number of optimization variables on the order $O[(NTNR)^2]$. We therefore conclude that the increase in computational complexity will not result in an infeasible problem size. Note that other overlap combinations are possible and would lead to different subvector lengths and different values of $K$. Once a solution is achieved, the signal vector can be formed as described above by (3.11) and used for detection purposes. In the following section, we describe a number of algorithms to optimize the problem with respect to the $w_k$'s.

### 3.3 Previous Work

The use of receive subarrays can be found throughout the literature [25, 32, 33, 34, 35]. In [25] it is noted that beamforming can first be performed on the subarrays to reduce the length of the data vector. A second beamformer can then be applied to the reduced vector, with the resulting overall beampattern being the product of the subarray and full beampatterns. For overlapping subarrays, a scaled version of the overall beampattern is expected. In [32, 33, 34, 35], various methods of beamforming and parameter identification are proposed to take advantage of this fact.

Numerous methods for receive beamforming can be found in Chapter 3 of [25], the most relevant of which is the method of Least Squares Error (LSE) pattern synthesis. Rather than matching the desired power, this method matches a desired beampattern,
Define the array beampattern as

\[ B(\theta) = w^H v(\theta), \]  

(3.14)

where \( w \) is the weight vector we wish to design, similar to the previous section. The square error is then

\[ \epsilon_1 = \sum_{l=1}^{L} |B_d(\theta_l) - w^H v(\theta_l)|^2. \]  

(3.15)

In this case, a closed-form minimum solution can be obtained quite readily

\[ w_{opt} = \left( \sum_{l=1}^{L} v(\theta_l)v^H(\theta_l) \right)^{-1} \sum_{l=1}^{L} v(\theta_l)B^*_d(\theta_l). \]  

(3.16)

Note that the matrix inversion is only possible in the case where the matrix \( \sum_{i=1}^{L} v(\theta_l)v^H(\theta_l) \) is full rank. This is often not the case and diagonal loading will be necessary when computing the inverse.

When subarrays are utilized, the error can be rewritten as

\[ \epsilon_2 = \sum_{l=1}^{L} \left| B_d(\theta_l) - \sum_{k=1}^{K} w_k^H v_k(\theta_l) \right|^2. \]  

(3.17)

Due to the use of the least squares cost function, the \( w_k \)'s can be optimized individually, yielding the solution

\[ w_{k, opt} = \left( \sum_{l=1}^{L} v_k(\theta_l)v_k^H(\theta_l) \right)^{-1} \sum_{l=1}^{L} \sum_{k=1}^{K} v_k(\theta_l)B^*_d(\theta_l). \]  

(3.18)

### 3.4 Receive Beampattern Matching

In this section, the proposed methods of receive beampattern matching are described. The methods are divided into two categories – one-pass algorithms and two-pass algo-
rithms – for reasons that will become clear upon description. We begin by presenting a method of solution for the given problem which uses the Gauss-Newton (GN) method for nonlinear least-squares problems [36]. We then describe a relaxation of the previously defined cost function which reformulates the problem as a SQP in order to create a weight matrix, similar to the method of transmit beampattern matching described in [17]. We then describe the Simple method for beampattern matching, and show how it can be combined with the other methods of solution, and itself, in order to achieve greater sidelobe suppression.

3.4.1 One-Pass Algorithms

Gauss-Newton Method for Nonlinear Least Squares

From (3.13), it is obvious that the problem fits the form of a least-squares problem as defined in [36]. The Gauss-Newton (GN) method is a well-known line search method for solving such problems iteratively [36, 37]. At each iteration, the method computes a search direction known as the Newton step, $p$, and moves in this direction by a step length, $t$. We begin by defining the residual, or difference between the desired and designed power for the location $\theta_l$ as

$$d_l(w) = P_r(\theta_l) - P_d(\theta_l),$$  \hspace{1cm} (3.19)

where

$$w = \begin{bmatrix} w_1 & w_2 & \ldots & w_K \end{bmatrix}^T \in \mathbb{C}^{KM \times 1}. \hspace{1cm} (3.20)$$

The difference vector of all grid points can be written as

$$d(w) = \begin{bmatrix} d_1(w) & d_2(w) & \ldots & d_L(w) \end{bmatrix}^T. \hspace{1cm} (3.21)$$
The problem can then be rewritten as

$$\min_w \frac{1}{L} d^T(w)d(w).$$

(3.22)

In this case, we can easily find $p$ by solving the system of equations described by [36]

$$J^HJp = -J^Hd(w),$$

(3.23)

where $J$ is the Jacobian matrix of $d(w)$ and has elements

$$J(i, j) = \frac{\partial d_i(w)}{\partial w_{ij}} = w^H_i v_i(\theta_l)v^*_j(\theta_l)$$

(3.24)

for $i = 1, \ldots, K$, and $j = 1, \ldots, M$. Note that $w_{ij}$ denotes the $j$th element of $w_i$, and $v_{ij}$ denotes the $j$th element of $v_i$ and $(\cdot)^*$ denotes the complex conjugate. We also scale both $J$ and $d(w)$ by a factor of $\sqrt{\frac{2}{L}}$ to account for the $L$ terms in the sum and the factor of 2 that results from taking the derivative of $d^T(w)d(w)$. The step size is computed using the backtracking line search described in [36]. Let $1$ denotes a vector of all ones, $c$ and $d$ be tuning parameters, and $\nabla$ denote the gradient. Algorithm 2 is then employed.

**Algorithm 2** Beamforming via the Gauss-Newton Method

```
input: $w = 1_{KM \times 1}$, error tolerance $\psi = 1 \times 10^{-6}$, tuning parameters $c = 1 \times 10^{-4}$, $d = 0.2$
initialize: $f(w) = \frac{1}{L}d^T(w)d(w)$
while $\|\nabla f(w)\| \geq \psi$ do
    $p = - (J^HJ)^{-1} J^Hd(w)$
    $t = 1$
    while $f(w + tp) \geq f(w) + ct [\nabla f(w)]^H p$ do
        $t = d \times t$
    end while
    $w = w + tp$
end while
```

This choice of initial guess generally results in faster convergence, though for
sufficiently small error tolerance, any feasible initial guess would work. Once the weight vector is obtained, it can be used to find the output data vector and power as described by (3.11) and (3.12).

**Semidefinite Quadratric Programming (SQP)**

The problem defined in (3.13) can also be transformed into a SQP, which can then be solved using public domain software such as [26]. We begin by rewriting (3.12) as

\[
P_r(\theta) = \sum_{k=1}^{K} v_k^H(\theta) w_k w_k^H v_k(\theta)
\]

\[
= \sum_{k=1}^{K} v_k^H(\theta) R_k v_k(\theta). \tag{3.25}
\]

Let vec\( (R_k) \) be the \( M^2 \times 1 \) vector obtained by stacking the columns of \( R_k \) on top of each other, and \( r_k \) be the \( M^2 \times 1 \) real-valued vector made from the diagonal elements of \( R_k \) and the real and imaginary parts of the upper triangle of \( R_k \). Following [17], it can be shown that

\[
\text{vec}(R_k) = Br_k, \quad k = 1, \ldots, K \tag{3.26}
\]

for \( B \) of size \( M^2 \times M^2 \) made up from the constants \((0, \pm j, \pm 1)\). Define

\[
h_k(\theta_l) = -[\left(v_k^T(\theta_l) \otimes v_k^H(\theta_l)\right) B]^T, \quad k = 1, \ldots, K. \tag{3.27}
\]

We can then rewrite (3.13) as

\[
J = \frac{1}{L} \sum_{l=1}^{L} \left| P_d(\theta_l) + \sum_{k=1}^{K} h_k^T(\theta_l) r_k \right|^2, \\
= \rho^T \Gamma \rho, \tag{3.28}
\]
where
\[ \rho = \begin{bmatrix} 1 & r_1^T & r_2^T & \ldots & r_K^T \end{bmatrix}^T \quad (3.29) \]
and
\[ \Gamma = \frac{1}{L} \sum_{l=1}^{L} \begin{bmatrix} P_d(\theta_l) \\ h_1(\theta_l) \\ \vdots \\ h_K(\theta_l) \end{bmatrix} \begin{bmatrix} P_d(\theta_l) & h_1^T(\theta_l) & \ldots & h_K^T(\theta_l) \end{bmatrix}. \quad (3.30) \]

Using this form, we arrive at the SQP
\[ \min_{\rho} \quad \rho^T \Gamma \rho \]
subject to \( R_k \geq 0, \ k = 1, \ldots, K. \quad (3.31) \]

Here the rank(\( R_k \)) = 1 constraint is relaxed to positive semidefiniteness, and therefore \( w \) cannot be recovered directly. Instead, we weight the signal by multiplying by \( R_k^{1/2} \).

This relaxation allows for performance benefits at the cost of increased data vector length. We then obtain the output
\[ \tilde{v}_k(\theta) = R_k^{1/2}v_k(\theta), \ k = 1, \ldots, K \quad (3.32) \]

and stack the resulting vectors to create a second virtual data vector of length \( KM \)
\[ \tilde{v}(\theta) = \begin{bmatrix} \tilde{v}_1^T(\theta) & \tilde{v}_2^T(\theta) & \ldots & \tilde{v}_K^T(\theta) \end{bmatrix}^T. \quad (3.33) \]

One added advantage to this method is that it allows the user to easily force the beampattern to a given constant, \( c \), by adding the constraint
\[ v_k(\theta_i)^H R_k v_k(\theta_i) = c, \ k = 1, \ldots, K, \quad (3.34) \]
for some desired location \( \theta_i \). If the user has a priori knowledge regarding the locations of interfering targets or direct-path interference, the beampattern nulls could be optimally placed and significant benefits in SINR could be obtained.

One problem with this technique is that the number of optimization variables can be extremely large. For example, in the case where \( N_T = N_R = 10 \) and \( K = 55 \), \( \rho \) is of length \( KM^2 + 1 = 111,376 \), requiring \( \mathbf{\Gamma} \) to have 12,404,613,376 elements and thus be computationally infeasible. However, this problem is easily overcome by optimizing each \( \mathbf{r}_k \) individually and combining, rather than optimizing all \( \mathbf{r}_k \)'s jointly. We show in Sec. 3.5.1, that while this does, in certain cases, result in a change in performance, the difference is minimal and we can still obtain performance advantages over the other methods described here.

**Simple Method**

The Simple method solves for a weight matrix, rather than a weight vector, in order to perform receive beampattern matching. The method multiplies the outer product of the data vector by the desired beampattern \( P_d(\theta_l) \) at each angle \( l \) under consideration. This method is also presented as an initial guess for the transmit beampattern matching method described in [20].

The weight matrix is defined as

\[
\mathbf{Q} = \frac{1}{N_T N_R} \sum_{l=1}^{L} P_d(\theta_l) \mathbf{v}(\theta_l) \mathbf{v}^H(\theta_l) \in \mathbb{C}^{N_T N_R \times N_T N_R}
\]  

(3.35)

We again multiply the square root of the weight matrix by the data vector \( \mathbf{v}(\theta) \) to obtain

\[
\hat{\mathbf{v}}(\theta) = \mathbf{Q}^{1/2} \mathbf{v}(\theta) \in \mathbb{C}^{N_T N_R \times 1}.
\]  

(3.36)
The received power after applying this weight matrix becomes

\[ P_r(\theta) = \mathbf{v}^H(\theta) \mathbf{Q} \mathbf{v}(\theta). \]  

Since we are approximating the solution, rather than solving the problem itself, it is now necessary to include a scaling factor, \( \alpha \), in order to match the magnitude of the given beampattern, such that

\[ \alpha P_r(\theta) \approx P_d(\theta). \]  

The scaling factor is described by the equation

\[ \alpha = \frac{\sum_{l=1}^{L} P_d(\theta_l)}{\sum_{l=1}^{L} P_r(\theta_l)}, \]

and the weight matrix \( \sqrt{\alpha} \mathbf{Q}^{1/2} \) can be multiplied by the signal in order to match the beampattern in both spatial quality and magnitude.

In the following sections, it is shown that this method can be combined with the previously mentioned GN and SQP methods, as well as itself, in order to further reduce sidelobe amplitudes. For all of the following methods, the scaling factor can be computed using (3.39) and applied in a manner similar to this section.

### 3.4.2 Two-Pass Algorithms

**Gauss-Newton-Simple Method**

In order to achieve greater sidelobe suppression, the Simple beampattern matching method can be combined with the GN, SQP, and itself. In the case of combining with the GN method, the use of the weight vector and partitioning gives the added advantage of shortening the data vector to be of length \( K \), which is always less than \( N_T N_R \). Given the resulting data vector of Section 3.4.1, which is defined by (3.11),
we apply the simple beampattern matching method to solve for the weight matrix

$$\tilde{Q} = \frac{1}{K} \sum_{l=1}^{L} P_d(\theta_l) \tilde{v}(\theta_l) \tilde{v}^H(\theta_l) \in \mathbb{C}^{K \times K}. \quad (3.40)$$

The output signal and resulting power are then described by the equations

$$\hat{v}(\theta) = \tilde{Q}^{1/2} \tilde{v}(\theta) \in \mathbb{C}^{K \times 1}, \quad (3.41)$$

and

$$P(\theta) = \tilde{v}^H(\theta) \tilde{Q} \tilde{v}(\theta). \quad (3.42)$$

Note that in Section 3.5.1 we compare the beampattern matching performance with the LSE-Simple method, which is formulated in the same manner as the Gauss-Newton-Simple but using \( w_k \) as defined by (3.18).

**SQP-Simple Method**

In a manner similar to the previous section, the SQP and Simple methods can be combined to reduce sidelobe levels. Given the output data vector of the SQP method defined in (3.33), we can apply the simple beampattern matching method to obtain the weight matrix

$$\tilde{Q} = \frac{1}{KM} \sum_{l=1}^{L} P_d(\theta_l) \tilde{v}(\theta_l) \tilde{v}^H(\theta_l) \in \mathbb{C}^{KM \times KM}, \quad (3.43)$$

and then multiply by \( \tilde{v}(\theta) \) to obtain the output signal

$$\hat{v}(\theta) = \tilde{Q}^{1/2} \tilde{v}(\theta) \in \mathbb{C}^{KM \times 1}. \quad (3.44)$$
The received power after applying this weight matrix becomes

\[ P(\theta) = \tilde{\mathbf{v}}^H(\theta) \tilde{\mathbf{Q}} \tilde{\mathbf{v}}(\theta). \]  

(3.45)

**Simple-Simple Method**

One final method of achieving greater sidelobe suppression relies on a combination of the Simple method with itself. We first partition the data signal as described in Section 3.2, and apply the Simple method to each partition of the signal vector individually to obtain

\[ \mathbf{Q}_k = \frac{1}{M} \sum_{l=1}^{L} P_d(\theta_l) \mathbf{v}_k(\theta_l) \mathbf{v}_k^H(\theta_l) \in \mathbb{C}^{M \times M}, \quad k = 1, ..., K. \]  

(3.46)

We multiply each signal partition by its respective weight matrix to obtain the output

\[ \hat{\mathbf{v}}_k(\theta) = \mathbf{Q}_k^{1/2} \mathbf{v}_k(\theta) \in \mathbb{C}^{M \times 1}, \quad k = 1, ..., K \]  

(3.47)

and stack the resulting vectors to create a new virtual data vector of length KM

\[ \hat{\mathbf{v}}(\theta) = \left[ \hat{\mathbf{v}}_1^T(\theta) \quad \hat{\mathbf{v}}_2^T(\theta) \quad ... \quad \hat{\mathbf{v}}_K^T(\theta) \right]^T. \]  

(3.48)

We apply the Simple method a second time, but now to the new data vector, \( \hat{\mathbf{v}} \), to obtain the weight matrix

\[ \hat{\mathbf{Q}} = \frac{1}{KM} \sum_{l=1}^{L} P_d(\theta_l) \hat{\mathbf{v}}(\theta_l) \hat{\mathbf{v}}^H(\theta_l) \in \mathbb{C}^{KM \times KM}. \]  

(3.49)

The output signal and resulting power are then described by the equations

\[ \hat{\mathbf{v}}(\theta) = \hat{\mathbf{Q}}^{1/2} \hat{\mathbf{v}}(\theta) \in \mathbb{C}^{KM \times 1} \]  

(3.50)
Figure 3.1: Beampattern-1 results formed by the GN method for various partition lengths.

\[ P(\theta) = \hat{v}^H(\theta) \hat{Q} \hat{v}(\theta). \] (3.51)

### 3.5 Simulation Results

In this section, simulation results are presented to demonstrate the performances of the proposed algorithms. In all of the simulations, a uniform linear array with half-wavelength interelement spacing at both the transmit and receive sides is assumed. For all of the simulations, \( N_T = N_R = 10 \) is used, except in the comparison of joint optimization and individual optimization using the SQP method, in which case we choose \( N_T = N_R = 5 \) to allow for manageable array sizes.

We begin by comparing the performance of the proposed methods for receive beampattern matching. Next, we simulate the resulting SINRs after applying beampattern matching at the receiver end. Finally, we show the effects of receive beampattern matching in a series of detection scenarios.
3.5.1 Beampattern Matching Performance

We begin by comparing the beampattern matching performance of the various methods in terms of suppression outside the ROI. We refer to the first three methods presented as one-pass methods and the final three methods as two-pass methods, since they achieve beamforming through the combination of two separate weighting schemes. The reflection coefficient $\beta(\theta) = 1$ for all values of $\theta$.

We first consider the beampattern as defined by

$$P_d(\theta) = \begin{cases} 1, & \theta \in [-30^\circ, 30^\circ] \\ 0, & \text{otherwise} \end{cases},$$

(3.52)

where $[a, b]$ denotes the set of direction angles between $a$ and $b$, including the endpoints. We refer to this as beampattern-1 in the following. Fig. 3.1 shows the simulation results to obtain this beampattern using the GN method with different values of $K$. It can be seen in the figure that the sidelobe levels are the lowest for $K = 80$. Besides the reduced sidelobe levels, this method has the advantage of re-
ducing the length of the data vector. This can be beneficial when applying target location and amplitude estimators such as the Capon or APES methods defined in Ch. 1 of [31], which rely on the inversion of the sample covariance matrix. Although not shown here, simulations show similar results for the SQP method, with the best performance achieved for $K = 90$. The difference in $K$ is due to the semidefinite relaxation on $R_k$. The relaxation alters the problem, yielding slightly different beampatterns and consequently lowest sidelobes for a different value of $K$. Fig. 3.2 shows the beampatterns obtained using the GN, SQP, and Simple methods, with the LSE beampattern shown for comparison. Each method’s respective best value of $K$ is used in order to fairly determine which method gives the best performance. Fig. 3.2 shows that the SQP method produces slightly lower sidelobes than the GN method; a result which is expected due to the relaxation of the rank($R_k$) = 1 constraint. However, this method results in a much larger signal vector length than that produced by the GN method (990 vs. 80 in this example). The Simple method results in lower sidelobes than the other one-pass methods, but does suffer some main lobe distortion toward the center of the ROI. The Simple method also provides a closed-form solution, rather than employing an iterative method, which requires far less computation time than the other methods presented. However, it is clearly seen that the LSE solution provides superior performance to all proposed methods, providing lower sidelobe levels and faster roll-off. This is likely due to the use of the beampattern, rather than the received power, in the problem formulation.

Fig. 3.3 shows the beampattern difference when optimizing the $R_k$’s jointly and individually with the SQP method for $N_T = N_R = 5$ and $K = 5, 7$, and 10. The resulting beampatterns are very similar, with differences depending on the value of $K$. The maximum difference, which occurs when $K = 5$, has a magnitude of 0.05. For this reason, we conclude that optimizing the $R_k$’s individually and then combining provides a suitable solution to the problem.
Figure 3.3: Beampattern difference when optimizing subvectors individually using the SQP method for various values of K under beampattern-1.

Figure 3.4: Comparison of beampatterns formed using the two-pass methods for beampattern-1.

We next show the simulation results obtained using the two-pass methods. Fig. 3.4 shows the resulting beampatterns formed by the two-pass methods for their respective best values of K. The GN-Simple and SQP-Simple methods show a reduction in sidelobe amplitudes of about 20 dB from their respective one-pass counterparts, but added main lobe distortion. For the GN-Simple method, the best performance is achieved with $K = 80$, as in the previous case. For the SQP-Simple case, choosing
Figure 3.5: Comparison of beampatterns formed using the two-pass methods for beampattern-2.

$K = 95$ performs slightly better than the previous best $K$ value of 90. Both methods perform better than the Simple method, which has the greatest suppression of all one-pass methods. The SQP-Simple method results in lower sidelobes than the GN-Simple method, but has slower roll-off and again results in a significantly larger data vector. The Simple-Simple method results in the greatest sidelobe reduction at the cost of slower roll-off and a significantly increased data vector length. The best performance for this method is achieved with the choice of $K = 38$. This is an important note because while the Simple-Simple method provides lower sidelobes than the other methods, the output signal vector length is much larger than that given by the SQP-Simple and GN-Simple methods. The LSE-Simple method provides the roll-off similar to the GN-Simple method, sidelobes comparable to the Simple-Simple, and a nearly flat passband region. It also results in a data vector length of 80, which is decreased from the initial size, making it the best performer of all methods shown.

In order to demonstrate the diversity of achievable beampatterns, we next consider
the scenario in which there are two separate ROIs, such that

\[ P_d(\theta) = \begin{cases} 
1, & \theta \in [-45^\circ, -35^\circ] \cup [35^\circ, 45^\circ] \\
0, & \text{otherwise} 
\end{cases}, \quad (3.53) \]

where \( \cup \) denotes the union of sets. This beampattern will be referred to as beampattern-2. The choice of ROIs is due to the fact that the SQP method is only capable of designing symmetric beampatterns. This is because the output weight matrix is completely real. However, the other methods do not suffer from this limitation and could be used in cases where ROIs are asymmetrically located. Fig. 3.5 shows the beampattern achieved by the two-pass methods. In this case, the Simple-Simple method provides lower sidelobes in the central region than the LSE-Simple method but provides a much slower roll-off, which may be undesirable in the case of nearby interfering targets.

The beampattern results shown here indicate that all proposed methods do an adequate job of matching the desired beampattern, though none of the proposed methods provide any performance benefits (either in sidelobe reduction, roll-off, or computational complexity) over the classical LSE method. In the following sections, the array benefits of beamforming are demonstrated through analysis of the proposed methods. Although not pictured, it is reasonable to assume that the LSE and LSE-Simple methods would provide benefits proportional to the beampattern matching performance demonstrated in this section.

### 3.5.2 SINR Performance

To show the advantages gained through receive beampattern matching, we analyze performance by comparing the SINRs. Consider the general case in which there exist \( T \) targets of interest and \( I \) interfering targets in the presence of independent and
identically distributed noise with variance $\sigma_n^2$. The SINR can then be expressed as

$$\text{SINR} = \frac{\sum_{t=1}^{T} P_r(\theta_t)}{\sum_{i=1}^{I} P_r(\theta_i) + \sigma_n^2}. \quad (3.54)$$

In the ideal case, where all interfering targets have been fully suppressed, the SINR reduces to the SNR. In the following, all targets of interest have reflection coefficients

Figure 3.6: SINRs under beampattern-1 with six interfering targets and three targets of interest for 3.6(a) one-pass methods and 3.6(b) two-pass methods.
Consider the scenario in which there are three targets of interest located at 0° and ±20°, and six interfering targets at ±60°, ±70°, and ±80°. Zero-mean circularly symmetric white Gaussian noise is added to the interference term. We apply the receive beampattern defined by (3.52), and calculate the SINR average over 1000 Monte Carlo trials. In this case, all interfering targets are within the suppressed region such that no method suffers due to poor roll-off. Fig. 3.6(a) shows a comparison of SINR vs. SNR for the one-pass methods, as well as the unaltered MIMO and best (SNR) cases. The SNR axis of the plot extends to 60 dB, which is unlikely to occur in a radar system. The axis is extended to this point to demonstrate that the SINR achieved by the proposed methods does not experience an indefinite linear increase, but rather flattens out at high SNR. Interestingly, the GN method performs better than the SQP. This is due to the interfering targets at ±80°, which experience roughly 23 dB of suppression via the SQP method and 31 dB of suppression via the GN method (Fig. 3.2). Similarly, Fig. 3.6(b) shows the performance of the two-pass methods with the MIMO and best cases. All methods result in a considerable gain in SINR over the MIMO case as well as the one-pass methods. The Simple-Simple method performs significantly better than the other two-pass methods due to its superior ability to reduce sidelobe levels. We can therefore conclude that the methods are suppressing interfering targets as expected.

We next consider the scenario in which the interfering targets are more closely located to the targets of interest. Here, we place the targets of interest at 0° and ±30°, and two interfering targets at ±35°. Fig. 3.7(a) shows the resulting SINRs for the two-pass methods. In this case, the GN-Simple method results in the best performance, since it has faster roll-off than the SQP-Simple and Simple-Simple methods. Fig. 3.7(b) shows an expanded view of the receive beampattern and targets for the GN-Simple and Simple-Simple methods. While the targets of interest receive roughly
the same suppression by the two methods, the GN-Simple method suppresses the interfering targets by 7 dB more than the Simple-Simple method, resulting in an improved SINR. Thus, we show that the location of targets can play a role in the choice of beampattern matching method.

As a final scenario, we consider receive beampattern-2 defined by (3.53). Here two targets of interest are placed at ±40°. Interfering targets are located at ±10°, ±20°,
and ±80°. Fig. 3.8 shows the resulting SINRs for selected methods. Although the Simple method has a slightly lower maximum sidelobe level than the GN-Simple, for most locations the GN-Simple provides lower sidelobes and thus results in a greater SINR gain. Again, the Simple-Simple method provides the largest gain but slower roll-off than the GN-Simple method, and in a scenario of closely spaced targets and interferers this method may not yield the best performance.

3.5.3 Detection Performance

To show the suppression ability gained by performing receive beampattern matching, we demonstrate detection using the Least Squares (LS) method under the two desired beampatterns presented. The application of more sophisticated detection methods, such as Capon and Amplitude-and-Phase Estimation (APES), can be derived in a similar manner. The received signal, $s$, is now composed of the targets of interest, interfering targets, and noise

$$s = \sum_{t=1}^{T} v(\theta_t) + \sum_{i=1}^{I} v(\theta_i) + \epsilon.$$  (3.55)
For a MIMO radar transmitting orthogonal signals, the reflection coefficient of a target present at angle $\theta_t$ can be found by minimizing the cost function

$$J = \min_\beta |s - \beta(\theta_t) a_T(\theta_t) \otimes a_R(\theta_t)|^2.$$  

(3.56)

The solution is easily obtained as

$$\beta(\theta_t) = \frac{(a_T(\theta_t) \otimes a_R(\theta_t))^H s}{(a_T(\theta_t) \otimes a_R(\theta_t))^H (a_T(\theta_t) \otimes a_R(\theta_t))} = \frac{1}{N_TR} (a_T(\theta_t) \otimes a_R(\theta_t))^H s.$$  

(3.57)

In the case where receive beampattern matching has been performed, the solution using a weight vector with $K = 1$ becomes

$$\beta(\theta_t) = \frac{[a_T(\theta_t) \otimes a_R(\theta_t)]^H \mathbf{w} \mathbf{w}^H s}{[a_T(\theta_t) \otimes a_R(\theta_t)]^H \mathbf{w} \mathbf{w}^H [a_T(\theta_t) \otimes a_R(\theta_t)]}.$$  

(3.58)

For the use of a weight matrix, e.g., the Simple and all two-pass methods, the value $\mathbf{w}$ can be replaced with $(\mathbf{R}^{1/2})^H$ in the above equation. After beamforming, the denominator is equivalent to the received power as described by (3.12). Ideally, this term is

$$P_r(\theta) = \begin{cases} 
1, & \theta \text{ in target region} \\
0, & \theta \text{ in interference region} 
\end{cases}$$  

(3.59)

which would result in instability. We therefore normalize by the received power, such that the reflection coefficient under a weight vector becomes

$$\beta(\theta_t) = (a_T(\theta_t) \otimes a_R(\theta_t))^H \mathbf{w} \mathbf{w}^H s.$$  

(3.60)
Figure 3.9: Detected targets after applying the GN and Simple methods with a ROI between ±30°. Targets and interferers are shown by dotted lines.

Figure 3.10: Target detection after applying the GN-Simple method for various partition lengths. Targets are shown as dotted lines.

Similarly, when a weight matrix is used,

$$\beta(\theta_t) = (a_T(\theta_t) \otimes a_R(\theta_t))^H (R^{1/2})^H (R^{1/2})s.$$  \hspace{1cm} (3.61)

For $K > 1$, we partition $[a_T(\theta_t) \otimes a_R(\theta_t)]$ and $s$ into $K$ subvectors and apply weighting as in (3.11) or (3.33).
Consider the first scenario of Section 3.5.2, in which there are three targets of interest within one single region. Again, the targets of interest have $\beta_t = 1$, while interfering targets have $\beta_i = 10$. Zero-mean circularly symmetric white Gaussian noise with variance $\sigma_n^2 = 0.01$ is added to the signal. Fig. 3.9 shows the resulting reflection coefficients after applying the GN and Simple methods of receive beampattern matching. All interfering targets are successfully suppressed by both methods, though the GN method results in significantly higher amplitudes in the suppressed region. The Simple method results in a significant downward bias in amplitude for the target located at $0^\circ$ due to the main lobe distortion that results from this method. The GN method does not experience the same distortion, but the target locations at $\pm 20^\circ$ are biased slightly inward due to the ripples in the main lobe. Similar distortions occur when using the other methods, with all methods showing the ability to completely suppress the interfering targets. Fig. 3.10 shows the reflection coefficients calculated by the GN-Simple method for several values of $K$. Due to the combination with the Simple method, the target at $0^\circ$ again experiences a downward bias in amplitude. Choosing $K$ less than 40 results in unreliable detection (false alarms). While the $K = 40$ case results in an upward bias in the amplitudes and inward biased location estimates, it can be seen that detection is still possible, even with a signal vector length that has been reduced by 60% of its initial value.

We now investigate the second scenario presented in Section 3.5.2, where there are two widely separated targets with six interferers spaced outside and between the targets. Fig. 3.11(a) shows the resulting reflection coefficients using the GN-Simple method for both the target-present and target-absent cases. The figure shows significant reflection coefficient amplitudes in the target-absent case. This occurs because of sidelobe leakage from interfering targets into the ROI, which causes the detection output to match the desired beampattern as shown in Section 3.5.1. The figure shows a difference in magnitude of roughly 0.4 between the two cases. The
Figure 3.11: Target detection after applying the GN-Simple method in the case of two separate ROIs, showing detected reflection coefficients when targets of interest are present and absent. 3.11(a) shows the results when the targets are spaced as described, while 3.11(b) shows the case where interfering targets have been moved farther from the ROIs. Targets are shown as dotted lines.

Simple and Simple-Simple methods result in greater differences, though all methods result in increased reflection coefficients in the ROIs regardless of target presence or absence. Fig. 3.11(b) shows the reflection coefficients in the case where the targets are spaced farther from the ROIs. In this case, the amplitudes within the ROIs for
the target absent case are significantly reduced. In all cases, the interfering targets are fully suppressed, and thresholding could be used to determine target presence or absence. More sophisticated detection methods could be a way to mitigate this problem in the future.

3.6 Conclusion

We have considered a series of solutions to the problem of receive beampattern matching for MIMO radar. It has been shown that partitioning the received data vector into $K$ subvectors improves performance for all methods described. In the presence of interfering targets, the application of receive beampattern matching results in a significant improvement in SINR. Detection scenarios using the LS method show that it is possible to suppress interfering targets and detect only targets located within a prespecified ROI. While the proposed methods do an adequate job of matching the desired beampattern, no performance benefits are obtained over the classical methods found in the literature.
Chapter 4

Concluding Remarks

4.1 Summary

Methods to improve the SINR of a MIMO radar system have been explored by utilizing signal processing methods at both the transmit and receive sides. In Chapter 2, a novel method of transmit beamforming based on the DFT was proposed. This method provides numerous computational advantages over methods currently in existence. A radar architecture for transmitting signals that match the desired covariance matrix was also presented. This architecture provides performance benefits but has the drawback of a high PAPR. In Chapter 3, several methods for beamforming at the receive side were proposed. These methods take advantage of the extended virtual data vector provided by MIMO radar, as well as the use of subarrays. While the proposed methods are capable of matching a desired beampattern, none provide any computational or performance benefits over classical methods.

4.2 Future Work

One natural extension of the work described here would be to apply adaptive signal design using the proposed transmit beamforming method. The computational efficiency of this method could allow for rapid beampattern design and provide per-
formance benefits. Another topic of interest in the MIMO community is that of STAP. Due to the virtual steering vector, the received signal covariance matrix can be extremely large, and thus methods for reducing the dimension or rank would be highly applicable. One final area of research is that of applying to compressed sensing to MIMO radar.
REFERENCES


5 Papers Submitted and Under Preparation